ANALYTICAL CONICS

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DEDICATED

TO

THE MEMORY

OF

MY BROTHER

HEMENDRANATH DATTA

1901—21

H. N. DATTA

PREFACE

In this book, I have tried to present the subject in a manner which, I think, is good for the student at least from the psychological point of view. It is based on my experience of students at more than one seat of learning.

Those who are already acquainted with the subject may find it interesting to note the use of the principle of transformation of co-ordinates in the discussion of the straight line (as well as of certain other topics), besides a number of other points interspersed throughout the book. They may also feel that the book has a character of its own.

The book is specially meant for students of under-graduate classes of Indian Universities. It does not deal with topics which are not generally required by them. It may be divided into two courses—articles not marked with an asterisk (or two) roughly forming the first.

In writing this book, I have derived considerable help from the well-known works on Conics (analytical or geometrical) by Salmon, Askwith, Sommerville, Smith, Mukhopadhyaya, Loney, Baker, Greenhill and others. I take this opportunity of expressing my grateful thanks to all of them for making it possible for me to write a small book on the subject. My thanks are also due to

the Manager and the Proprietor of the Sakha Press for printing the book off in a very short time inspite of various preoccupations.

If this book serves the purpose for which I have written it for the use of the students, if it be of any real service to them, then at least one of the dreams of my life will have been realized.

For any criticisms, suggestions, or corrections (of misprints or actual errors), I shall be grateful.

ALIGARH
August 11, 1931.

H. N. DATTA.

CONTENTS

ANALYTICAL CONICS

CHAPTER I.

INTRODUCTORY.

1.

Preliminary Remarks.

Geometry provides us with the endless task of studying the properties of each and every figure when it stands by itself or in relation with any other figure (or figures). In Analytical Geometry, we study the subject by reducing every problem of Geometry to one in Analysis (Algebra, Trigonometry, etc.).

The student who has learnt to draw the graphs of algebraic expressions (or to prove geometrical theorems corresponding to algebraic formulæ) may have already felt the possibility of studying Geometry with the help of Algebra. But though Vieta (1540—1603) and others had applied Algebra to Geometry before Descartes (1596—1650), the latter was the first to introduce into Geometry an analytical method based on the notions of constants and variables. He gave a method[1] (known to students acquainted with Graphs) of representing the position of a point by means of algebraic symbols. This together with the already known result that a single equation in two variables admits of an infinite number of

[1] Published in his *Discours sur la methode etc.* in 1637.

solutions gave Descartes the necessary start, and is mainly responsible for the whole of Analytical Geometry (including that of the Conic Sections) as hitherto studied.

The system of representation discovered by Descartes is known as the *Cartesian* system. It does not, however, provide us with the only method for the study of Geometry with the help of Algebra, Trigonometry, etc. At present, we have other recognized methods for such studies—the method of *Polar* co-ordinates, the method of *Trilinear* co-ordinates, etc. But the Cartesian method is the first, and by far the most effective method discovered so far for such studies.

The student who has noticed the advantage of applying Algebra to certain types of arithmetical problems will very likely find himself similarly well-placed in his study of Geometry with the help of Analysis. He may feel the presence of some *guiding methods* which help him, in many cases, to make a reasonable start towards the solution of a problem, if not also to actually complete it. This is the real blessing of Analytical Geometry as a whole. It appears to be an advantage over the *apparently abrupt procedure* one has to follow in his study of what may be called "the School Geometry." But, "Is this new method a universal one applicable to all conceivable questions in Geometry?" That it is not so is clear from the fact that all questions in Analysis are *not necessarily* soluble. In many cases, a considerable amount of the difficulties in analysis can be overcome by choosing the fundamental quantities (the axes, etc.) in a special manner, or by making a suitable beginning.[2] But it should be remembered that the present-day mathematician admits the necessity of the existence of many methods, many geometries—each appealing to certain topics, each adding to the subject of Geometry as a whole. He does so because of his knowledge of a very prolonged competition among the advocates of different methods of studying Geometry—a healthy rivalry which not only enriched the subject as a whole, but had made the geometers realize that the greatest strength lay *not* in the suppression of some methods in preference to others but in the comparative study of them all.

The beginner should try to clear up his ideas on the subject-matter of this book by drawing figures for himself as often as he finds them necessary. This will help him in removing much of the difficulties he is likely to experience before getting habituated to this new method of studying Geometry. He should, moreover, try to cultivate the habit of *distinguishing the principles from the analytical details.* For, an imperfect knowledge of the ordinary analytical processes (solution, elimination, etc.) oftentimes leads to a great confusion of thought in a subject like Analytical Geometry; and the habit of *dividing a problem into a number of shorter ones* will not only help him in understanding the principles underlying various types of problems, but will also give him a chance of knowing (and therefore, of removing) some of his defects in Analysis. [In this connection, it may be

[2] Here comes in the *personal factor* the development of the efficiency of which in case of the average student is one of the principal aims of this book.

advantageous for him to note that the analytical treatment of a geometrical theorem may be regarded as consisting of the following parts :—

(i) The geometrical statement of the data (that is, the conditions under which the theorem is to hold good).

(ii) The expression of the given conditions in an analytical language.

(iii) The application of suitable analytical processes (solution, elimination, etc.) to the results of (ii).[3]

(iv) The interpretation of the result of these operations for the purpose of arriving at some geometrical conclusion.}

It is hoped that the *general arrangement of the topics* discussed in this book together with the *footnotes* (explanatory or otherwise) and *worked-out examples* (specially meant for illustrative purposes) will help the beginner in acquiring a clear idea of the subject without the help of a teacher.

2.

CO-ORDINATES.

Co-ordinates[4] are quantities that determine the position of a geometrical element (point, line, etc.) with reference to some known geometric system (a set of lines, a set of points, etc.). The term being a *relative* one, it is clear that the co-ordinates of the same element will be

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[3] At this stage, the *geometrical concept* may be entirely lost sight of.

[4] The term "*co-ordinate*" was first used by Leibnitz in 1694.

different for different reference-systems even when we restrict ourselves to the same mode of representation.

2·1. Cartesian co-ordinates.

In the Cartesian system, the geometrical element is a *point*; and when the space considered is a *plane* (as in this book), the reference-system consists of two intersecting straight lines called the *axes of co-ordinates*. The point of intersection is called the *origin*. The axes are said to be *rectangular or oblique* according as these lines do or do not intersect at right angles.[5]

The axes divide the plane containing them into four parts called the *quadrants*.

(*i*) *When the axes are rectangular* :—

Let the axes XOX', YOY' be rectangular; and let their positive directions be as indicated in fig. 1.[6] Also let P be any point in their plane, and in the quadrant XOY. Draw PM parallel to OY and PN parallel to OX. Then, by measuring the distances OM and ON, we can easily get an idea as to how the point P is situated with regard to the axes. Conversely, if the distances OM and ON are known, we can find the point

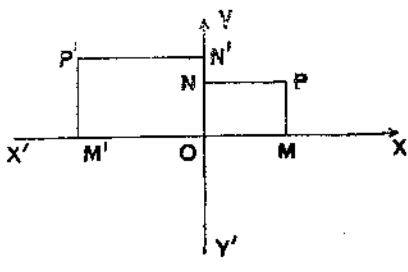

FIG. 1

[5] *In what follows, the axes will be supposed to be rectangular unless the contrary is explicitly stated.*

[6] In what follows, the *positive* direction of the axis of x will be taken towards the *right*, and that of the axis of y in the *upward* direction as in fig. 1.

P by constructing the parallelogram (rectangle) $OMPN$ on OM and ON as adjacent sides, provided that the quadrant in which P lies is known. If, however, distances measured from O towards the directions $\overline{OX}$, $\overline{OY}$ are taken with a positive sign (and distances measured in the direction $\overline{OX'}$, $\overline{OY'}$ are taken with a *negative sign*), then it is *not* necessary to state the quadrant in which P must lie. For, to find the point R' given by $OM' = -3$ and $ON' = +4$, we must (according to the above convention) take $OM' = 3$ units along OX' and $ON' = 4$ units along OY, and complete the parallelogram (rectangle) $OM'P'N'$ (fig. 1); and so on.

In fig. 1, let $OM = x$ and $ON = y$. Then x and y (attention being paid to their magnitudes as well as signs) are called the *cartesian co-ordinates of P with reference to the axes OX, OY*. x is called the *abscissa* of P, and y its *ordinate*. Such a point P will be denoted symbolically as the point (x, y), the abscissa with its proper sign being put first. Thus, the point whose abscissa $= -3$ and ordinate $= 4$ will be denoted as the point $(-3, 4)$.

* (*ii*) *When the axes are oblique :—*

In this case, the co-ordinates are measured in exactly the same manner as before; but the angle XOY being not equal to a right angle, parallelograms like $OMPN$ cease to be rectangles any longer.

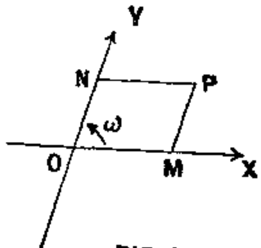

FIG. 2

The angle XOY is usually denoted by ω.[7]

[7] Articles (or portions thereof) marked with an asterisk (*) deal with *oblique* co-ordinates, and may be omitted at the first reading.

2·2. Polar co-ordinates.

In the polar system, the geometrical element is a point; and the reference-system (for points lying in a given plane as in this book) consists of a straight line OX (in the same plane) and a point O on the straight line. OX is called the *initial line*, and O the *pole*.

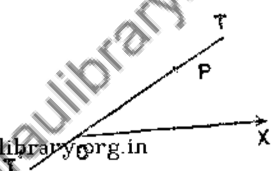

FIG. 3

It is easy to see that the position of a point P is determined *uniquely*, if the angle XOT and the distance OP are known (in magnitude and sense of measurement). These quantities are, therefore, the co-ordinates of the point P in this system. The angle XOT is called the *vectorial* angle, and is measured positively in the anti-clockwise direction (and *negatively* in the clockwise direction) from the initial line OX. It is usually denoted by θ. Unless otherwise stated, it will be supposed to be positive and less than two right angles.

The distance OP is called the *radius vector*, and is measured positively along OT (and *negatively* along OT'). It is usually denoted by r.

To locate the position of a point P whose polar co-ordinates are r and θ,[8] it is sufficient to draw the angle XOT equal to θ (in magnitude and sign), and then measure

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Those marked with two asterisks (**) are of a harder nature, and do not call for serious attention at the first reading.

[8] Such a point will be symbolically denoted as the point (r, θ), the radius vector with its proper sign being put first.

a length OP equal to r (along OT or OT' according as r is positive or negative).[*]

2·3. Both the signs of the radius vector are necessary.

So far as isolated points are concerned, the admission of only one sign to the radius vector is sufficient. For, the point $(-r, \theta)$ can be represented equally well by $(r, \theta+\pi)$ by adjusting the vectorial angle. But when we have to deal with points whose polar co-ordinates satisfy a given equation, we may have to admit *both* the signs. For instance, the point $\left(-1, \frac{\pi}{6}\right)$ satisfies the equations $\frac{1}{r}=1+4\sin\theta$; but though $\left(1, \frac{\pi}{6}\right)$ represents the same point (without any change of the reference-system), the co-ordinates $\left(1, \frac{\pi}{6}\right)$ do not satisfy it. On the other hand, the point $\left(\frac{6}{7}, \frac{\pi}{3}\right)$ satisfies the equation $\frac{1}{r}=1+\frac{1}{3}\cos\theta$; but though $\left(-\frac{6}{7}, \pi+\frac{\pi}{3}\right)$ represents the same point, the coordinates $\left(-\frac{6}{7}, \pi+\frac{\pi}{3}\right)$ do not satisfy it. Hence *the need of both the signs to the radius vector.*

Examples.

1. Plot the following points (referred to the same axes):—$(3, 0)$, $(-4, 5)$, $(\pm 3, \pm 4)$, $(a, -2a)$ and $(a+1, a-1)$.

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[*] We can, of course, locate the position of P by drawing a circle with centre O and radius equal to r, and then noting the point where the arm OT (or OT', if r is *negative*) of the vectorial angle XOT cuts the circle.

2. Plot the following points (referred to the same system) :—

$$\left(\frac{1}{2},\ 60^\circ\right),\ \left(-2, \frac{\pi}{4}\right),\ \left(1,\ \frac{\pi}{2}\right),\ \left(-a, \theta\right) \text{ and } \left(a, 2\theta\right).$$

3. Are the points ($\frac{1}{2}$, 60°) and ($-\frac{1}{2}$, 240°) really different?

Do both of them satisfy the equation $\frac{1}{r} = 5 + 4 \cos \theta$?

3.

RELATIONS BETWEEN CARTESIAN & POLAR CO-ORDINATES OF A POINT.

3·1. A special set, the axes being rectangular.

Let the *x-axis* of the Cartesian system be taken as the *initial line* of the Polar system; and let the *origin* be the *pole*. Also let (x, y) and (r, θ) be the Cartesian and Polar co-ordinates of any point P with reference to these systems.

Draw $PM \perp$ to Ox (fig. 4). Then,

$x = OM = r \cos \theta$, ...(1)
and $y = MP = r \sin \theta$....(2)

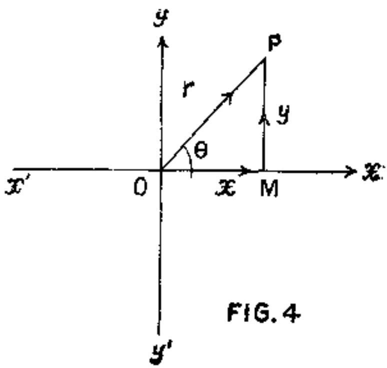

FIG. 4

These equations give the Cartesian co-ordinates of a point explicitly in terms of its

Polar co-ordinates, the reference-systems being chosen as above.[10–11]

* 3·2. A special set, the axes being oblique.

Take the x-axis as the initial line, and the origin as the pole. Also let P be any point whose co-ordinates referred to the two systems are (x, y) and (r, θ). Draw $PM \parallel$ to Oy and $PQ \perp$ to Ox (fig. 5). Then, if ω be the angle between the axes Ox and Oy, we have

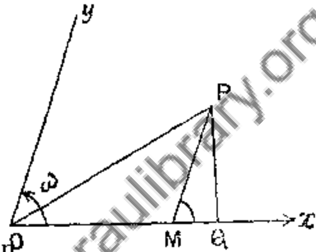

FIG. 5

$$OQ = OM + MQ.$$

$$\left.\begin{aligned} \therefore \quad r\cos\theta &= x + y\cos\omega. \\ \text{Again,}\quad r\sin\theta &= y\sin\omega, \\ \text{each being equal to } PQ. \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

[10] Unless otherwise stated, the reference-systems (in all cases of change from the Cartesian to the Polar system, and vice versa) will be supposed to have been chosen as in Art. 3·1.

The process of changing from one reference-system to another of a similar or different type is known as the *transformation of co-ordinates*. It is a great instrument of mathematical analysis. A judicious use of the process enables us to simplify matters in many cases—a fact which the beginner may realize in course of his progress in the subject.

[11] To express r and θ explicitly in terms of x and y, solve these equations for r and θ, or proceed independently from fig. 4.

Applying the first of these two methods, we have

$$r^2 = x^2 + y^2, \text{ by squaring and adding (1) and (2).}$$

$$\therefore \quad r = \pm\sqrt{x^2 + y^2}. \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

Again, dividing (2) by (1), we have $\tan\theta = \dfrac{y}{x}$. Hence, $\theta = \tan^{-1}\dfrac{y}{x}$.

These equations give r and θ *explicitly* in terms of x and y; but they do *not* determine them *uniquely*. In fact, we have two values for each of these quantities for every set of values of x and y; and but for the restriction imposed on θ (Art. 2·2), we would have an *infinite* number of values of θ for every set of values of x and y.

These equations give the formulæ of transformation in this case.[12]

Examples.

1. What are the polar co-ordinates of the points whose Cartesian co-ordinates are given in Ex. 1, Art. 2?

2. What are the Cartesian co-ordinates of the points whose polar Co-ordinates are given in Ex. 2, Art. 2?

3. Transform the equation $x^2+y^2-2ax=0$ to one in polar co-ordinates.

4. Transform the equation $r^2=2a \cos \theta$ to one in Cartesian co-ordinates.

4.

RELATION BETWEEN THE CO-ORDINATES OF A POINT IN ONE CARTESIAN SYSTEM AND THOSE IN ANOTHER.

4·1. When the new axes are parallel to the old axes and pass through a different origin.

Let Ox, Oy be the old axes and O_1x_1, O_1y_1 be the new axes; also let (h, k) be the point O_1 with reference to the old axes. Then, if P be any point whose co-ordinates referred to the old and new axes are (x, y) and (x_1, y_1) respectively, we have

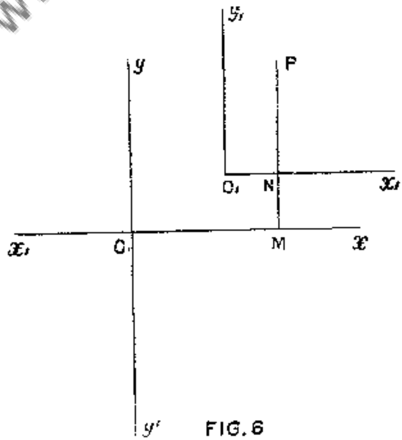

FIG. 6

[12] These relations are evidently more complicated than those of Art. 3·1. They are seldom required for practical use.

Relations between the two systems can be similarly obtained whenever their relative positions are known.

$$x = OM = h + x_1, \quad \ldots \quad \ldots \quad (1)$$
$$y = MP = k + y_1. \quad \ldots \quad \ldots \quad (2)$$

These equations give the old co-ordinates of a point in terms of its new co-ordinates (h, k being known quantities), and vice versa.[13-14] [They can, therefore, be used in transforming an equation in (x, y) into one in (x_1, y_1), and vice versa.][15]

4·2. When the directions of the axes are changed without changing the origin, both systems being rectangular.

Let the new axis Ox_1 be inclined at $\angle\theta$ to the old axis Ox. Then, the angle $yOy_1 = \theta$.

Also let P be any point whose co-ordinates are (x, y) and (x_1, y_1) with reference to the old and new axes respectively.

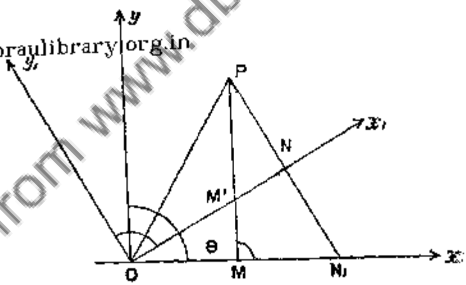

FIG. 7

[13] As the quantities h, k, x_1, y_1 may be supposed to include their proper signs, these formulæ of transformation are quite *general* (that is, *not restricted* to configurations of the type indicated in fig. 6). This can be easily seen by obtaining the formulæ of transformation with the help of any other figure, and then comparing them with those obtained above.

Similar remarks apply to the formulæ of transformation obtained in Arts. 3·1 and 3·2 with the help of figures 4 and 5 respectively.

[14] The old axes may be made to coincide with the new axes by moving O to O_1 without changing the directions of the axes Ox, Oy during the motion. Because of this fact, the axes are said to be *translated* from the first to the second position, and the equations (1) and (2) are called the *equations of translation* for the axes.

[15] It may be noted that the equations of translation for the axes would be of the same forms as in (1) & (2) even if the axes were *oblique*.

Draw $PM'M$, PNN_1 $\perp^s$ to Ox and Ox_1 respectively, and join OP (fig 7).

Now, since $ONPO$ is a closed figure, the orthogonal projection of OP on Ox = the algebraic sum of the orthogonal projections of ON and NP on the same line. [16] Hence,

$$OP \cos xOP = ON \cos xOx_1 + NP \cos xN_1P,\text{[17]}$$

or, $$x = x_1 \cos\theta - y_1 \sin\theta. \quad \ldots \quad \ldots \quad (3)$$

Again, taking the projections of the same figure on MP, we see that the projection of OP on MP = the algebraic sum of the projections of ON and NP on the same line. Hence,

$$OP \sin xOP = ON \cos x_1M'P + NP \cos NPM,$$

$$\text{or } x = x_1 \sin\theta + y_1 \cos\theta. \quad \ldots \quad \ldots \quad (4)$$

This is clear from the fact that the equations were obtained without making any use of the property that distinguishes rectangular axes from oblique axes.

In what follows, it will be a good exercise for the student to find out which of the results obtained for rectangular axes would be true for *oblique axes* as well.

[16] The student must be already aware of the meaning of the expression "the orthogonal projection of AB on CD".

He can easily see from a diagram that the algebraic sum of the orthogonal projections of the sides of a closed polygon (taken in order) on any straight line is zero. Hence the above result, $ONPO$ being the closed figure in this case.

[17] In such cases of algebraic measurement, we must measure angles, etc. in the same manner. Hence, having taken xOP as the angle which OP makes with Ox, we must take xN_1P as the angle which NP makes with Ox (the sign being +, or − according as the measurement is made in the anticlockwise or clockwise direction from Ox).

These equations give the old co-ordinates of any point P in terms of its new co-ordinates (θ being the same for all positions of P).[18, 19]

4·3. The most general case, both systems being rectangular.

From Arts. 4·1 and 4·2, it is clear that in the most general case when the axes are moved in any manner, they can be transferred from their old positions to their *new* positions by a *translation and a rotation*, the two axes being regarded as a rigidly-connected system during the motion.[20] The formulæ of transformation in this case are

$$\left.\begin{aligned} x &= h + x_1 \cos\theta - y_1 \sin\theta \\ \text{and } y &= k + x_1 \sin\theta + y_1 \cos\theta, \end{aligned}\right\} \qquad \ldots \qquad \ldots (5)$$

where (x, y) and (x_1, y_1) are the co-ordinates of any point P with reference to the old and new axes respectively (h, k and θ being the same as before).

For, suppose the system is *translated* first. Then, we have

$$x = h + x_2$$
$$\text{and } y = k + y_2,$$

where (x_2, y_2) are the co-ordinates of P with reference to the axes in the translated position.

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[18] As the terms old and new are *relative*, it is clear that the new co-ordinates can be expressed in terms of the old co-ordinates by *interchanging x and x_1, y* and y_1, and writing $-\theta$ for θ in the equations (3) and (4), $-\theta$ being the angle which Ox makes with Ox_1 since $+\theta$ is the angle which Ox_1 makes with Ox. These relations can also be obtained by solving (3) and (4) for x_1 and y_1, or by proceeding independently from fig. 7.

[19] In this case, the old axes may be made to coincide with the new axes by turning them through an angle θ (in the anticlockwise direction) about O. For this reason, (3) and (4) are generally called *the equations for rotating the axes.*

[20] A rigidly connected system is one in which there is no relative displacement of the points of the system.

Next, let these latter axes be rotated through an angle θ in the anticlockwise direction so that they may coincide with the new axes O_1x_1, O_1y_1.[21] Then, we have

$$x_2 = x_1 \cos\theta - y_1 \sin\theta$$

and $y_2 = x_1 \sin\theta + y_1 \cos\theta$.

Combining these two sets of equations, we have the equations (5).

4·31. Degree of an equation is unaltered by such a transformation.

In the most general case of transformation from one system of rectangular axes to another, an equation in x, y is transformed into one in x_1, y_1 by writing expressions of the first degree in x_1, y_1 for both x and y (Art. 4·3).[22] Hence, if the terms of the highest degree in the given equation be x^n, y^n, etc., those in the transformed equation will be of the types (an expression of the first degree)^n, (another expression of the 1st degree)^n, etc. But these latter expressions *cannot* contain any term of degree higher than n.[23] Hence, *such a transformation cannot raise the degree of the transformed equation.* Neither can the degree be *lowered* by such a transformation. For, by transforming the new equation thus obtained back again to the old axes, we must arrive at the original equation in x, y; and if the first trans-

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[21] We say anticlockwise because of the configuration as sketched in fig. 7. There is no loss of generality in such a statement.

[22] An equation in x, y is said to be of the nth degree, if the sum of the indices of x and y in any single term is n, the equation having been previously reduced so that the indices are the smallest possible integers. Thus, $x^2+y^2-2x+1=0$ and $x^{\frac{1}{2}}+y^{\frac{1}{2}}=1$ are both of the second degree in x, y.

[23] Our knowledge of the *Multinomial theorem* tells us so.

formation had lowered the degree, the second must increase it—a result contrary to what has been just proved. Hence, *the degree of an equation is unaltered by any change of rectangular axes.*

*4·4. The most general case, the axes being oblique.

Let ω be the angle between the old axes, and ω' the angle between the new axes; and let the co-ordinates of the new origin O_1 with reference to the old axes be (h, k). Then, to obtain the formulæ of transformation, we proceed thus :—

First, let the old axes be *translated* so that O may coincide with O_1.

In this case, we have

$$\left.\begin{aligned} x &= h + x_2 \\ \text{and } y &= k + y_2, \end{aligned}\right\} \quad \dots \quad \dots \quad (6)$$

where the co-ordinates of any point P before and after *translation* are (x, y) and (x_2, y_2) respectively.

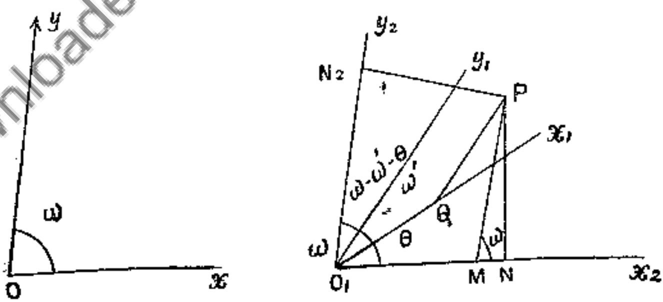

FIG. 8

Next, let the new axes O_1x_1, O_1y_1 be such that the x_2-axis (which is the x-axis in the translated position) may coincide with O_1x_1 when the former is rotated through an angle θ in the *anticlockwise* direction, and the y_2-axis may coincide with O_1y_1 when O_1y_2 is rotated through an angle $\omega-\omega'-\theta$ in the *clockwise* direction (fig. 8). [24]

Then, by drawing $PN \perp$ to O_1x_2, $PN_2 \perp$ to O_1y_2, $PQ \parallel$ to O_1y_1 and $PM \parallel$ to O_1y_2 (fig. 8), and equating to zero the algebraic sum of the orthogonal projections of the sides of the polygon O_1MPQO_1 on PN and PN_2 separately, we have

$$\left.\begin{aligned} y_2 \sin\omega &= x_1 \sin\theta + y_1 \sin(\omega'+\theta) \\ \text{and } x_2 \sin\omega &= x_1 \sin(\omega-\theta) + y_1 \sin(\omega-\omega'-\theta), \end{aligned}\right\} \ldots \ldots \quad (7)$$

where (x_1, y_1) are the co-ordinates of P in the new system. [25]

Combining these two sets of equations, we have the desired formulæ of transformation.

***4·41. Degree of an equation is unaltered by such a transformation.**

As the expressions for x and y are still of the first degree in x_1 and y_1, and as the deduction of the result of Art. 4·31 depends on the *degree of* (*but not on the nature of*) *the equivalent expressions, it is clear that the degree of an equation is unaltered by any change of Cartesian axes (rectangular or oblique).*

Examples.

1. What does the equation $x^2+y^2-2ax=0$ become when the origin is transferred to (a, o) without changing the directions of the axes?

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[24] Note that there is *no loss of generality* in the drawing of the figure according to the statement made in this paragraph.

[25] It may be noted that the left-hand side of the first equation of (7) represents the algebraic sum of the projections of O_1M and

2. What does the equation $x^2+4xy+y^2=9$ become when the axes are rotated through an angle of $45°$ in the anticlockwise direction?

3. Simplify the equation $x^2+4y^2-4x-8y+7=0$ by translating the axes.

[*Solution.* Putting $x=x_1+h$ and $y=y_1+k$, we have
$$x_1^2+4y_1^2-2x_1(2-h)-8y_1(1-k)-4h-8k+h^2+4k^2+7=0. \quad \ldots \quad \ldots \quad (8)$$

Now, choosing h and k so that the co-efficients of x_1 and y_1 may vanish separately, we have
$$h=2,\ k=1.$$

Substituting these values of h and k in (8), we get
$$x_1^2+4y_1^2-1=0$$
as the simplified equation.][26-29]

**4. What is the angle through which the axes must be turned in order that, in the transformed form of

---

MP on *PN*, and that its right-hand side represents the algebraic sum of the projections O_1Q and QP on the same line. [The second equation has been obtained by taking their projections on PN_2].

[26] This shows that an equation in x, y can be put in a more or less compact form by transforming it from one set of Cartesian axes to another. In fact, it may be possible for us to remove as many as three terms from an equation in x, y by choosing h, k and θ properly.

[27] The procedure to be followed in all such cases is this :—

Put $x=x_1+h$ and $y=y_1+k$, and equate to zero the coefficients of x_1 and y_1 in the transformed equation. Then, solve the two equations thus obtained for h and k, and substitute their values in the transformed equation.

[28] From (8), it is clear that we cannot choose h or k so as to make the coefficient of any term of the second degree vanish. This can be easily seen to be true in case of any equation of the second degree in x, y. Hence, a *translation of the axes cannot remove any term of the second degree (from an equation of the same degree).*

[29] The suffixes in x_1 and y_1 might be suppressed in case there is no confusion between the new x, y and the old x, y. The beginner

the equation $ax^2+2hxy+by^2=1$, the xy-term may not appear?

[*Solution.* Let the axes be turned through an angle θ in the *anticlockwise* direction. Then, from Art. 4·2, we have

$$x=x_1\cos\theta-y_1\sin\theta$$

and $$y=x_1\sin\theta+y_1\cos\theta.$$

Substituting these expressions for x and y in the above equation, we have

$$a(x_1\cos\theta-y_1\sin\theta)^2+2h(x_1\cos\theta-y_1\sin\theta)$$
$$\times(x_1\sin\theta+y_1\cos\theta)+b(x_1\sin\theta+y_1\cos\theta)^2=1. \quad \dots \quad (9)$$

The coefficient of x_1y_1 in this case is

$$-2a\cos\theta\sin\theta+2h(\cos^2\theta-\sin^2\theta)+2b\cos\theta\sin\theta.$$

Equating it to zero and simplifying, we have

$$\tan 2\theta=\frac{2h}{a-b}.$$

$$\therefore \qquad \theta=\tfrac{1}{2}\tan^{-1}\frac{2h}{a-b}, \qquad \dots \qquad (10)$$

which gives the angle through which the axes must be turned in the anticlockwise direction so that there may be no term in x_1y_1 in the transformed equation.] [30] [31]

should, however, guard himself against any error arising out of the use of this sort of *notations* in which the same letters are used in different senses in different parts of the same problem.

[30] The tangent of an angle may have any value from $+\infty$ to $-\infty$. Hence, a real value of θ is possible in all cases, a, h, b being real. Hence, the xy-term can always be removed from an equation of the above type (by rotating the axes).

[31] It is not that any other term of the second degree cannot be removed by rotating the axes; but it can be seen that, unlike the case of the xy-term, a real value of θ does not necessarily exist for all real values of a, h and b. Thus, the x^2-term can be removed by choosing θ so that $b\tan^2\theta+2h\tan\theta+a=0$ (i.e., $b\tan\theta=-h\pm\sqrt{h^2-ab}$). Hence, $\tan\theta$ (and consequently, θ) will be real when and only when $h^2>ab$.

A term of the first degree can always be removed by choosing θ properly; but it is inexpedient to do so.

**5. Remove the xy-term from $3x^2+4xy+y^2+5x=7$ by rotating the axes.

**6. Simplify the equation $x^2+4xy+y^2+2x-3y=5$ by removing the terms in x, y and xy. [32]

[*Hints.* Remove the terms in x and y by translating the axes (Ex. 3), and then remove the xy-term from the newly-formed equation by rotating the axes (Ex. 4).] [33]

**7. Remove the terms in x and x^2 from the equation of Ex. 6.

8. Transform the equations $x=0$, $y=k$ and $Ax+By=0$ to the form $ax+by+c=0$ in each case.

[*Hints.* All the equations can be transformed into the given form by applying the most general formulæ obtained in Art. 4·3.] [34]

**9. Show that x^2+y^2 becomes $x_1{}^2+y_1{}^2$ by rotating the axes through any angle without changing the origin.

**10. If $ax^2+2hxy+by^2$ becomes $a_1x_1{}^2+2h_1x_1y_1+b_1y_1{}^2$ by a rotation of axes without change of origin, show that

$$ab-h^2=a_1b_1-h_1{}^2$$

and

$$a+b=a_1+b_1.$$ [35]

[32] As $h^2>ab$ in this case, we could simplify the equation by removing the terms in x, y and x^2 (or y^2).

[33] As the axes can be translated or rotated quite independently of each other, it is clear that the final result does not depend on the order in which these operations are performed.

[34] While the most general transformation is necessary in the first case, a suitable rotation of axes is sufficient for the second, and a translation for the third.

[35] If any function of the coefficients of a *polynomial* in x, y be equal to the same function of the coefficients of the polynomial obtained by a transformation of co-ordinates, it is called its *invariant*.

In this case, $ab-h^2$ and $a+b$ are both *invariants* of $ax^2+2hxy+by^2$.

[*Solution*. We know that x^2+y^2 becomes $x_1{}^2+y_1{}^2$ by a rotation of axes without change of origin. Hence,

$$ax^2+2hxy+by^2+\lambda(x^2+y^2) \text{ becomes}$$
$$a_1x_1{}^2+2h_1x_1y_1+b_1y_1{}^2+\lambda(x_1{}^2+y_1{}^2) \text{ in this case.}$$

Now, if *any value of* λ *makes the first expression a perfect square, the same value of* λ *must make the second expression a perfect square.*[36-37]

But the first expression is a perfect square,

if $\quad (a+\lambda)(b+\lambda)=h^2,$

or if $\quad ab-h^2+(a+b)\lambda+\lambda^2=0;$

and the second expression becomes a perfect square,

if $\quad (a_1+\lambda)(b_1+\lambda)=h_1{}^2,$

or if $\quad a_1b_1-h^2{}_1+(a_1+b_1)\lambda+\lambda^2=0.$

Hence, these quadratic equations in λ must have the same roots.

$\therefore \quad ab-h^2=a_1b_1-h_1{}^2$

and $\quad a+b=a_1+b_1.$][38]

*11. Show that $x^2+2xy\cos\omega+y^2$ becomes $x_1{}^2+2x_1y_1\cos\omega'+y_1{}^2$ by any change of oblique axes without change of origin.

*12. If $ax^2+2hxy+by^2$ becomes $a_1x_1{}^2+2h_1x_1y_1+b_1y_1{}^2$ by any change of oblique axes without change of origin, show that

[36] The equations of transformation being linear functions of the coordinates of each system, the *squared* form of an expression cannot be changed by such transformations. For, if the first expression be $=(lx+my)^2$, then the second expression must be $=\{l(x_1\cos\theta-y_1\sin\theta)+m(x_1\sin\theta+y_1\cos\theta)\}^2$, a perfect square.

[37] A more direct method (though a rather lengthy one) of procedure would be to verify the results by applying the formulæ of transformation of Art. 4·2, and equating the coefficients of like terms in the given and transformed equations in x_1, y_1.

[38] This method of proving the results is due to Boole.

$$\frac{a+b-2h\cos\omega}{\sin^2\omega}=\frac{a_1+b_1-2h_1\cos\omega'}{\sin^2\omega'}$$

and
$$\frac{ab-h^2}{\sin^2\omega}=\frac{a_1b_1-h_1^2}{\sin^2\omega'}.$$

[*Hints.* Proceed as in Ex. 10 using the results of Ex. 11 instead of those of Ex. 9.]

*13. If $x=lx_1+my_1$ and $y=l_1x_1+m_1y_1$ be the formulæ of transformation to a new set of axes without change of origin, prove that

$$mm_1(l^2+l_1^2-1)=ll_1(m^2+m_1^2-1).$$

[*Hints.* We have $x^2+2xy\cos\omega+y^2=(lx_1+my_1)^2+$ etc $=x_1^2+2x_1y_1\cos\omega'+y_1^2$. Now, equate the coefficients of x_1^2 and y_1^2 on both sides, and eliminate $\cos\omega$.]

5.

DISTANCE BETWEEN TWO POINTS WHOSE CO-ORDINATES ARE GIVEN.

5·1. Expression for the distance, the axes being rectangular.

Let $P\equiv(x_1, y_1)$ and $Q\equiv(x_2, y_2)$ be the two points. Draw PM, QN parallels to OY and $PR \parallel$ to OX (Fig. 9).

Now, since PQR is a $\triangle$ right-angled at R, we have

$$PQ^2=PR^2+RQ^2.$$

But $\overline{PR}=x_2-x_1$

and $\overline{RQ}=y_2-y_1$.

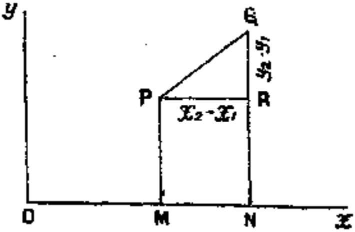

FIG. 9

Hence, $PQ^2=(x_2-x_1)^2+(y_2-y_1)^2$.[39] ... ... (1)

[39] Hence, the square of the distance of the point $P\equiv(x,y)$ from the origin $=x^2+y^2$ (the axes being rectangular). [Note that the result

*5·2. Expression for the distance, the axes being oblique.

When the axes are *oblique*, the triangle PQR is not a right-angled one. Hence, we have

$$\overline{PQ}^2 = \overline{PR}^2 + \overline{RQ}^2 - 2\overline{PR}.\overline{RQ} \times \cos PRQ.$$

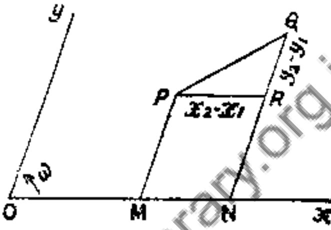

FIG. 10

But $\overline{PR} = x_2 - x_1$,

$\overline{RQ} = y_2 - y_1$,

and the angle $PRQ = \pi - \omega$ (ω being the angle between the axes) as in fig. 10.[40] Hence,

$$PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + 2(x_2 - x_1)(y_2 - y_1)\cos\omega. \quad \ldots \quad \ldots \quad (2)$$

5·3. Expression for the distance in polar co-ordinates.

Let the x-axis be taken as the initial line, and the origin as the pole; also let (r_1, θ_1) and (r_2, θ_2) be the polar co-ordinates of P and Q whose Cartesian co-ordinates are (x_1, y_1) and (x_2, y_2) respectively. Then, from (1), we have

$$PQ^2 = (r_2 \cos\theta_2 - r_1 \cos\theta_1)^2 + (r_2 \sin\theta_2 - r_1 \sin\theta_1)^2$$
$$= r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_2 - \theta_1). \quad \ldots \quad \ldots \quad (3)$$

5·31. Alternative Method.

From the triangle OPQ, we have

$$PQ^2 = OP^2 + OQ^2 - 2.OP.OQ. \cos POQ.$$
$$= r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_2 - \theta_1),$$

of Ex. 9 of Art. 4 can be obtained from the consideration that both $x^2 + y^2$ and $x_1^2 + y_1^2$ are equal to OP^2.]

[40] The angle PRQ is not always equal to $\pi - \omega$; but taking account of the signs of the quantities involved, it will be seen that PQ is given by (2) in all cases.

since $\cos POQ = \cos(\theta_2 - \theta_1)$ for all positions of P and Q.[41]

Examples.

1. Find the distance between the following pairs of points :—

(1, 2) and (2, 1) ; $(a \cos\theta, a \sin\theta)$ and $(a \cos\phi, a \sin\phi)$
(1, 60°) and (− 2, 45°) ; $(at^2, 2at)$ and $(a\mu^2, 2a\mu)$.

2. Show that the points (0, 0), (2, 0) and $(1, \sqrt{3})$ form an equilateral triangle.

3. Prove that the points (−1, 1), (5, 3), (11, 9) and (5, 7) are at the vertices of a parallelogram ; and find the lengths of the diagonals.

[*Hints.* As the points may not be given in the order in which the vertices are situated, it is best to proceed as follows :—

Plot the points roughly to get an idea of the figure ; and then show that the opposite sides of the quadrilateral so formed are equal (a parallelogram being a quadrilateral whose opposite sides are equal). The relative positions of the vertices being known, the lengths of the diagonals can be easily found by (1).]

4. Show that the points (0, 0), (1, 1), (0, 2) and (−1, 1) are at the vertices of a square.

---

[41] By varying the relative positions of P and Q, and trying to obtain an expression for the distance in each case, it will be found that the cosine of the angle POQ is always $= \cos(\theta_2 - \theta_1)$.

(It may be noted that the method of Art. 5·3 requires a knowledge of the formulæ of transformation of Art. 3·1, but does not require any discussion of this kind).

6.

AREA OF A TRIANGLE, THE CO-ORDINATES OF THE VERTICES BEING GIVEN.

6·1. Sign of an area.

Let $P \equiv (r_1, \theta_1)$ and $Q \equiv (r_2, \theta_2)$ be any two points; and let us consider the triangle OPQ.

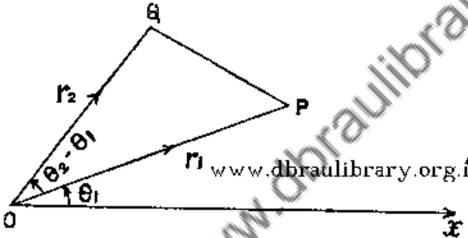

FIG. 11

Its area $= \frac{1}{2}.OP.\ OQ \sin POQ = \frac{1}{2} r_1 r_2 \sin(\theta_2 - \theta_1)$, as is clear from fig. 11.

Interchanging P and Q (which means an interchange of the co-ordinates of P and Q), we get $\frac{1}{2} r_2 r_1 \sin(\theta_1 - \theta_2)$ as the area of the same triangle.

Now, as these expressions (which ought to have been identically equal) are of different signs, *the area of a triangle must be regarded as capable of having a sign (from the analytical point of view).*

In what follows, *we shall take the area of a △OPQ with a positive or a negative sign according as, in going round the triangle from O to* **P** *and then to Q, we find the area always to our left or always to our right.* [42]

[42] It can be easily seen that the area of the $\triangle OPQ = \frac{1}{2} r_1 r_2 \times \sin(\theta_2 - \theta_1)$ in *all* cases in which in going round from O to P and then

6·2. Expression for the area of any triangle in polar co-ordinates.

Let $P \equiv (r_1, \theta_1)$, $Q \equiv (r_2, \theta_2)$ and $R \equiv (r_3, \theta_3)$ be the vertices of any triangle PQR. Then, taking into consideration the signs of areas, we have (fig. 12)

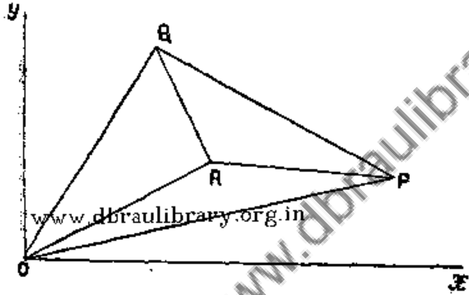

FIG. 12

$$\triangle\ PQR = \triangle\ OPQ + \triangle\ OQR + \triangle ORP$$

$$= \pm\tfrac{1}{2}\{r_1 r_2 \sin(\theta_2 - \theta_1) - r_2 r_3 \sin(\theta_2 - \theta_3) - r_3 r_1 \sin(\theta_3 - \theta_1)\}.$$

$$= \pm\tfrac{1}{2}\{r_1 r_2 \sin(\theta_2 - \theta_1) + r_3 r_1 \sin(\theta_1 - \theta_3) + r_2 r_3 \sin(\theta_3 - \theta_2)\}, \quad \ldots \quad (1)$$

+ or − sign being taken according as, in going round the triangle from P to Q and then to R, we find the area always to our left or always to our right.

6·3. Expression for the area, the axes being rectangular.

Take the initial line as the x-axis and the pole as the origin; and let $P \equiv (r_1, \theta_1) \equiv (x_1, y_1)$, $Q \equiv (r_2, \theta_2) \equiv (x_2, y_2)$ and $R \equiv (r_3, \theta_3) \equiv (x_3, y_3)$. Then, from (1), we have

---

to Q, the area is to our *left*, and $= \frac{1}{2} r_1 r_2 \sin(\theta_1 - \theta_2)$ when the area is to our *right*. So, the question of *signs* can be settled.

$\triangle PQR = \pm\frac{1}{2}\{r_1 r_2 (\sin\theta_2\cos\theta_1 - \sin\theta_1\cos\theta_2) +$ two similar expressions$\}$.

$= \pm\frac{1}{2}\{(x_1 y_2 - x_2 y_1) + (x_3 y_1 - x_1 y_3) + (x_2 y_3 - x_3 y_2)\}$, by Art 4'1.[43-44]

$$\therefore \triangle PQR = \pm\frac{1}{2}\begin{vmatrix} x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \\ x_3, & y_3, & 1 \end{vmatrix}, \quad \ldots \quad \ldots \quad (2)$$

as can be easily verified by expanding the determinant.[45-46]

---

[43] This result can be obtained directly as follows :—

$\triangle PQR =$ trapezium $MR -$ trapezium $NR -$ trapezium MQ. (fig. 13).

$$= \tfrac{1}{2}\{(x_3 - x_1)(y_1 + y_3) - (x_3 - x_2)(y_3 + y_2) - (x_2 - x_1)(y_2 + y_1)\}$$

$$= \tfrac{1}{2}\{x_3y_1 - x_1y_3 + x_2y_3 - x_3y_2 + x_1y_2 - x_2y_1\} \quad \ldots \quad \ldots \quad (3)$$

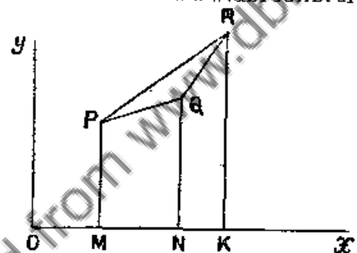

FIG 13

[44] Interchanging P and Q (which means an interchange of the coordinates of P and Q), we have

$$\triangle PQR = \tfrac{1}{2}\{x_3y_2 - x_2y_3 + x_1y_3 - x_3y_1 + x_2y_1 - x_1y_2\}.$$

$=$ minus the expression obtained in (3).

This shows (without the help of polar co-ordinates) that the area of a triangle is *capable of having a sign* (in analysis).

[45] $\begin{vmatrix} a, & b \\ c, & d \end{vmatrix}$ is a symbolical way of representing the expression $ad - bc$. It is called a *determinant of the second order, and the letters* a, b, c, d are called its *constituents*. Similarly,

$\begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix}$ stands for $a_1\begin{vmatrix} b_2, & c_2 \\ b_3, & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2, & c_2 \\ a_3, & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2, & b_2 \\ a_3, & b_3 \end{vmatrix}$,

* 6·4. Expression for the area, the axes being oblique.

Choosing the axes as in **Art.** 3·2, and applying to (1) the formulæ of transformation obtained there, we have

$$\triangle\ PQR = \pm \tfrac{1}{2}\ \sin\omega \begin{vmatrix} x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \\ x_3, & y_3, & 1 \end{vmatrix}$$

as the area of the triangle in this case.

Examples.

1. Find the area of the triangle whose angular points are $(1, -2)$, $(1, 2)$ and $(-3, 1)$.

2. Show that the area of the triangle whose vertices are $(a\ \mu^2_r,\ 2a\ \mu_r)$, where $r = 1, 2, 3$ respectively is given by $a^2\ (\mu_1 - \mu_2)(\mu_2 - \mu_3)(\mu_3 - \mu_1)$.

3. Find the area of the quadrilateral whose angular points are $(2, 1)$, $(1, 3)$, $(-5, 1)$ and $(-1, -2)$.[47]

---

or $a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$ and is called a *determinant of the third order*; and so on.

In connection with the expansion of a determinant of the third order, it may be noted that the determinants of the second order which are factors of a_1, $-b_1$, and c_1 are obtained from the original determinant by scoring out the row and column containing a_1, b_1 and c_1 respectively. An extension of this idea will help us in expanding a determinant of any order.

[46] The following properties of a determinant can be easily verified :—

(i) If two rows or columns are interchanged, the determinant changes its sign.

(ii) If two rows or columns are identical, the determinant vanishes.

For further information, see Hall and Knight's *Higher Algebra*, or Burnside and Panton's *Theory of Equations*, Vol. II.

[47] The order of the points do not necessarily indicate the order of the vertices of the quadrilateral.

4. Find the area of the pentagon whose angular points are (1, 3), (2, 4), (−3, 2), (−4, −2) and (0, −3).

5. *Find the condition that three points whose co-ordinates are given may be collinear.*

[*Solution.* Let $P \equiv (x_1, y_1)$, $Q \equiv (x_2, y_2)$ and $R \equiv (x_3, y_3)$ be the three points. They will lie in one straight line *when and only when* the area of the triangle formed by them is zero. Hence, by Art. 6·3, we have

$$\begin{vmatrix} x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \\ y_3, & y_3, & 1 \end{vmatrix} = 0, \quad \dots \quad \dots (4)$$

which gives the condition for collinearity of the three points.][48]

7.

THE POINT WHICH DIVIDES IN A GIVEN RATIO THE LINE JOINING TWO GIVEN POINTS.

7·1. When the axes are rectangular.

Let $P \equiv (x_1, y_1)$ and $Q \equiv (x_2, y_2)$ be the two given points, and $R \equiv (x, y)$ a point on PQ such that $PR : RQ = m_1 : m_2$.

Draw PM, QN, RL parallels to Oy, and $PT \parallel Ox$ (Fig. 14).

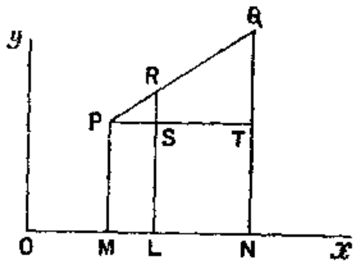

FIG. 14

Then, since the $\triangle PRS$, PQT are similar, we have

$$\frac{PS}{PT} = \frac{PR}{PQ} = \frac{m_1}{m_1 + m_2}.$$

But $PS = x - x_1$ and $PT = x_2 - x_1$,

[48] It can be easily seen that the same condition holds good even when the axes are *oblique*.

$$\therefore \quad \frac{x - x_1}{x_2 - x_1} = \frac{m_1}{m_1 + m_2},$$

or $$x = \frac{m_2 x_1 + m_1 x_2}{m_1 + m_2}, \quad \ldots \quad \ldots \quad (1)$$

which gives the abscissa of the point R.

Similarly, it can be seen that the ordinate of R is given by

$$y = \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2}.[49] \quad \ldots \quad \ldots \quad (2)$$

7·11. Distinction between internal and external division.

Now, $\frac{PR}{RQ}$ = *a positive* or *a negative* quantity according as the point R divides PQ *internally* or *externally*.[50] So, if the point R divides PQ *externally*, the quantities m_1, m_2 must be taken with different signs. In case of an *internal* division, both of them must be taken with the same sign.

***7·2. When the axes are oblique.**

In this case, it can be easily seen that the co-ordinates of R have the same forms as before.[51]

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[49] These results are due to Joachimsthal (1818–1861).

[50] In algebraic measurements, if PR is taken *positively*, then RQ must be taken negatively when R is outside the finite segment PQ. Hence the result.

[51] In each case, the co-ordinates of the middle point of PQ will be

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right), \text{ since } m_1 = m_2 = 1.$$

Examples.

1. Find the co-ordinates of the point which divides the straight line joining the points (2, 3) and (3, 2) in the ratio of $1 : -5$.

2. In what ratio is the straight line joining the points (1,3) and (2, 9) divided *internally* by the point (2, 5), and *externally* by the point (3, 15) ?

3. Find the centroid of the triangle the coordinates of whose angular points are given (the axes being rectangular or oblique).

[*Solution.* The centroid is the point of intersection of the medians and divides them in the ratio of 2 : 1. Hence, to find the centroid G of a triangle ABC, all that we have to do is to find the middle point D of a side BC, and then find the point G which divides AD internally in the ratio of 2 : 1. So, if (x_1, y_1), (x_2, y_2). (x_3, y_3) be the co-ordinates of A, B, C respectively, the point D is $\{\frac{1}{2}(x_2 + x_3), \frac{1}{2}(y_2 + y_3)\}$, and the point G is

$$\left\{\frac{2.\frac{1}{2}(x_2 + x_3) + 1.x_1}{2+1}, \frac{2.\frac{1}{2}(y_2 + y_3) + 1.y}{2+1}\right\}$$

or
$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right).]$$

4. In any triangle ABC, show that

$AB^2 + AC^2 = 2(AD^2 + DC^2)$, D being the middle point of the side BC.

[*Hints.* Let $A \equiv (x_1, y_1)$, $B \equiv (x_2, y_2)$ and $C \equiv (x_3, y_3)$, the axes being rectangular. Now, verify the truth of the theorem by finding the co-ordinates of D as well as expressions for AB, AC, AD and DC.][52]

5. If G be the centroid of any triangle ABC, then $AB^2+BC^2+CA^2=3(GA^2+GB^2+GC^2)$.

8.

EQUATIONS AND LOCI. [53]

8·1. Geometrical significance of an equation between the co-ordinates of a point.[54]

When an equation between the co-ordinates x, y of a point is given, it is generally possible to find an *infinite* number of points whose co-ordinates satisfy it. [55] The assemblage of these points whose co-ordinates (and

[52] When a geometrical theorem is given, the result can be easily verified by means of analysis. A suitable choice of axes will certainly simplify our labour in this connection; but once the principle has been properly understood, it will not be difficult to settle the details connected with the rest.

[53] A group of points obeying some common law is said to form a *locus* (the group being supposed to include all such points and no others). From this, it is clear that *a locus is not necessarily a curve* (by which we mean a continuously drawn line—straight or curved). In this book, we shall be mainly concerned with problems in which the one is the other.

[54] For this purpose, it is sufficient to discuss equations in rectangular co-ordinates only. For, equations in oblique or polar co-ordinates can be transformed into equations in rectangular co-ordinates; and a change of the reference-system can change the form of the equation, but not the *geometrical nature* of the locus represented by it. (The whole thing would be meaningless otherwise).

[55] In the equation $x+y+5=0$, we may assign to x any value we please, and find the corresponding value of y by solving the resulting equation. This will help us in plotting an infinite number of points. On the other hand, it is easy to see that the equation $x^2+y^2+1=0$ is not satisfied by any real values of x and y. Hence, an equation in x, y is *not necessarily* satisfied by the co-ordinates of any real point. Moreover, since the equation $x^2+y^2=0$ is satisfied by the co-ordinates of only one real point, *viz.* (0, 0), it is clear that an equation may be satisfied by the co-ordinates of only a finite number of points. Hence, we *cannot* say that an infinite number of real points can be found in all such cases.

those of no other points) satisfy the given equation forms what is called its *geometrical significance or locus.* [56–58] Thus, the equation $(x-1)^2+(y-2)^2=9$ shows that the distance of the point (x, y) from the fixed point $(1, 2)$ is always $=3$. Hence, it is clear that the equation is satisfied by the co-ordinates of any point on the circle whose centre is at the point $(1, 2)$ and radius $=3$, and by *no other points.* This circle is, therefore, the locus represented by the given equation. [59]

8·11. Note on equations of the form $S+\lambda S'=0$.

Let $S=0$ and $S'=0$ be two equations in x, y. Then, if the co-ordinates of any point satisfy each of the equations $S=0$ and $S'=0$, its coordinates will automatically satisfy the equation $S+\lambda S'=0$, where λ is a constant. Hence, it is clear that $S+\lambda S'=0$ represents a locus through all the

[56] In cases like $x+y+5=0$, the locus becomes what we call a *curve* (real). In cases like $x^2+y^2+1=0$ where the locus is altogether imaginary, we shall say that the locus is an *imaginary curve.* And in cases like $x^2+y^2=0$; the locus consists of a finite number of real points. [In the present case, the curve consists of two imaginary branches passing through the real point (0, 0).]

[57] All points which satisfy the equation $x+y=0$ also satisfy the equation $(x+y)(x-y)=0$; but all points which satisfy the second equation do not necessarily satisfy the first. Hence, the geometrical significance of the first equation cannot be the same as that of the second.

[58] The method of finding the co-ordinates of a point which satisfy a given equation consists in putting a suitable value for x (or y), and then solving the resulting equation in y (or x) to find the value of the second co-ordinate corresponding to an assigned value of the first. By repeating this process as often as desired, we may find the co-ordinates of any number of points.

[59] We are not yet in a position to state the nature of a curve from its equation. It is a pleasant coincidence of circumstances that has helped us in naming the curve represented by the equation $(x-1)^2+(y-2)^2=9$ at this stage.

points (besides other points) common to the loci represented by $S=0$ and $S'=0$ taken separately.[60]

8·2. Any two equations of the first degree in *x* and *y* taken together represent only one point.

Solving $x+y=5$ and $3x+2y=13$ as simultaneous equations, we have $x=3$, $y=2$.

Similarly, by solving any two equations of the first degree in x, y as simultaneous equations, we shall get one (and only one) set of values for x and y. Hence, two equations of the first degree in x, y taken together represent only one point.[61-62]

****8·3. Analytical representation of a group of points.**

We cannot give a collective analytical representation to a group of points taken at random; but if the points are situated according to some known law, it *may* be possible for us to represent them collectively by means of an equation or inequality.[63] Thus :—

(i) The group of points which lie along the arc of a circle can be represented by means of an equation. For, if $C \equiv (x_1, y_1)$ be the centre of the circle and r its radius, then any pt. $P \equiv (x, y)$ which lies on the circle must satisfy the relation

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[60] In fact, if without transgressing the laws of algebra, we obtain an equation from two given equations, it will represent a locus passing through the points common to the loci represented by the given equations.

[61] In other words, there is only one point common to the loci represented by these equations taken separately.

[62] Similarly, it can be easily seen that there are two points (real or imaginary) common to the loci represented by two equations of the first and second degrees respectively; and so on.

[63] In analytical geometry, we are mainly concerned with problems in which every geometrical condition satisfied by a point leads us to an *equation* (or *equations*—not *inequalities*) involving the co-ordinates of the point.

$$(x-x_1)^2+(y-y_1)^2=r^2. \quad \ldots \quad \ldots \quad (1)$$

Moreover, no point which does not lie on this circle can satisfy this relation. Because of these facts, we say that the group of points which lie on the given circle is represented by the equation (1) [or that (1) is the equation of the given circle].

(ii) If $R\equiv(x, y)$ be any point within the circumference of a circle having its centre at the origin and radius equal to r, it must satisfy the inequality $x^2+y^2<r^2$. For, $x^2+y^2=$ the square of the distance of R from the origin; and this must be less than the square of the radius of the circle in order that the point may lie within the circumference of the circle. Moreover, no point which does not lie within the circle can make $x^2+y^2<r^2$. Hence, the group of points lying within this circle is represented by the inequality $x^2+y^2<r^2$.

(iii) On the other hand, *if we are asked to give a collective representation to the group of points lying within a rectangle, then a single equation or inequality is not sufficient for our purpose.* As a matter of fact, if we take the axes to be parallel to the sides of the rectangle, then four inequalities of the type $a<x<b$, $c<y<d$ are required to give a collective representation to this group of points.

Examples.

1. Does the equation $x^2+y^2+1=0$ represent any real locus?

2. Trace the locus represented by the equation $3x-2y-8=0$.

[*Solution.* (i) Solving for y, we have $y=\frac{3}{2}x-4$.

(ii) Assuming $x = 2, 4, 6$, etc. successively and computing y, we get $y = -1, 2, 5$, etc. respectively. Then, plotting the points $(2, -1)$, $(4, 2)$, $(6, 5)$, etc., we get a rough idea as to the nature of the required locus. (The locus *appears* to be a straight line as in fig. 15).][64]

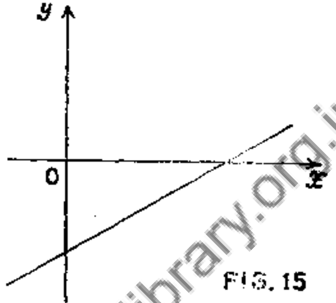

FIG. 15

3. Trace the locus represented by the equation $x^2 + y^2 = 4$.

4. Trace the locus of the equation $x^2 + 4y^2 = 24$.

[*Solution.* (i) The equation is already in its simplified form. So there is no need of trying to simplify it as in Ex. 3 (or 4 Art.) of 4.

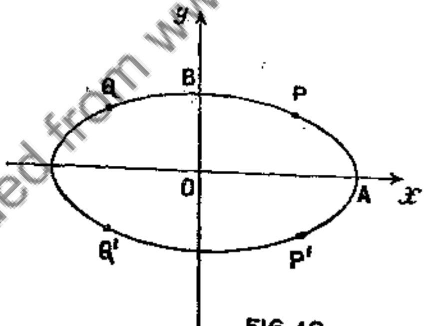

FIG. 16

(ii) The curve is symmetrical about the axes of x and y.

(iii) It is symmetrical about the origin.

(iv) Solving for x, we have $x = \pm\sqrt{24 - 4y^2}$. So, in order that x may be real $24 - 4y^2$ must be *positive*. Hence, y must lie between $+\sqrt{6}$ and $-\sqrt{6}$.

[64] The fundamental process of plotting a large number of points is good enough in ordinary cases. In case of equations of the second

Again, solving for y, we have $2y = \pm\sqrt{24 - x^2}$. Hence, in order that y may be real, x must lie between $+\sqrt{24}$ and $-\sqrt{24}$. Consequently, the locus is a closed *curve*.

(v) The origin is not on the curve as (0, 0) does not satisfy it. It cuts the x-axis where $x = \pm\sqrt{24}$, and the y-axis where $y = \pm\sqrt{6}$.

(vi) Assuming $x = 2, 4, \ldots$ successively, and computing y, we get $y = \pm\sqrt{5}, \pm\sqrt{2}, \ldots\ldots\ldots$ respectively.

(vii) Considerations of symmetry tell us that the points $(-2, +\sqrt{5})$ and $(-2, -\sqrt{5})$ lie on the locus, since the points $(2, +\sqrt{5})$ and $(2, -\sqrt{5})$ are on the locus. Repeating this consideration with regard to each point found in (vi), we get a large number of points on the locus.

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degree in x, y, it is very convenient to trace the locus represented by an equation after simplifying it with the help of Arts. 4·1 and 4·2. To avoid errors of drawing, the following questions should be answered before drawing a free-hand curve through the points plotted in any manner (a knowledge of the nature of the locus and other connected points being certainly very useful in this case) :—

(1) Is the curve symmetrical about any axis? (A curve is symmetrical about the axis $x=0$ if its equation is unaltered by writing $-x$ for x).

(2) Is it symmetrical about the origin? (A curve is symmetrical about the origin if its equation is unaltered by writing $-x$ for x and $-y$ for y simultaneously).

(3) Is the curve a closed one, or does it extend to infinity?

[*For this purpose*, solve the equation for x and determine the limits (if any) of the values of y so that x may be *real*; and then repeat the process by solving for y. *If the intervals between the limits be finite in both cases, the curve is a closed one.* If not, it will extend to infinity in a manner that may be seen from other considerations.]

(4) *Is the origin a point on the curve? What are the points of intersection of the curve and the axes?*

[The first part can be easily verified. To find where it cuts the y-axis, put $x=0$ in the given equation and then solve the resulting equation in y. Similarly, the points where it cuts the x-axis can be easily found.]

(viii) Plot all these points, and draw a free-hand curve through them (Fig. 16).]

5. Draw the graph of the equation $y^2 = 4ax$.

[*Hints.* Proceed as in Ex. 4 as far as possible. As y may have any value for *real* values of x, it will be noticed that the curve is *not* closed one (fig. 17).]

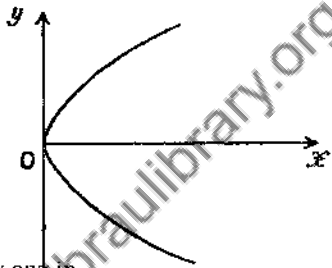

FIG. 17

6. Trace the loci represented by the following equations :—

(i) $x^2 - 4y^2 = 16$; (ii) $x^2 - 2x + y^2 = 0$;

(iii) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$; (iv) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

7. Find the locus of a point which moves so that its distances from two given points may be in the ratio of $m : n$.

[*Solution.* Let $P \equiv (x_1, y_1)$, $Q \equiv (x_2, y_2)$ be the fixed points and $R \equiv (x, y)$ be the moving point. Then,

$$PR^2 = (x - x_1)^2 + (y - y_1)^2$$

and $$RQ^2 = (x - x_2)^2 + (y - y_2)^2.$$

By the condition of the problem, we have

$$\frac{PR}{RQ} = \frac{m}{n}.$$

Hence (expressing the geometrical condition analytically), we have

$$n\sqrt{(x - x_1)^2 + (y - y_1)^2} = m\sqrt{(x - x_2)^2 + (y - y_2)^2}$$

$$\therefore \quad n^2\{(x - x_1)^2 + (y - y_1)^2\} = m^2\{(x - x_2)^2 + (y - y_2)^2\},$$

after rationalization.

This is a relation between the co-ordinates of any point on the locus, and is *free* from any other variable quantity. Hence, it is the equation of the required locus.]

8. Find the locus of a point which moves so that its distance from the point $(0, a)$ bears a constant ratio to its distance from the straight line $x+a=0$.

[*Solution*. Let (x, y) be any point on the locus. Then, by the condition of the problem, we have

$$x^2+(y-a)^2=(x+a)^2.$$

$$\therefore \qquad y^2=2a(x+y),$$

This (being a relation between the co-ordinates of any point on the locus and some known quantities) is the equation of the locus.] [65]

9. Find the locus of a point which moves so that the sum of the squares of its distances from three given points may be = 8.

10. Show that the locus of a point which moves so that the difference of its distances from the points $(m, 0)$ $(-m, 0)$ may be equal to $2c$ is given by

$$\frac{x^2}{c^2}-\frac{y^2}{m^2-c^2}=1.$$

*[?]11. Find the locus of the vertex of a triangle which has a given base and a given area. [66]

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[65] The beginner may have already felt that Analytical Geometry is specially suitable for the solution of locus problems. (It is the connection between an equation and a locus that forms the basis of Analytical Geometry).

[66] Beginners should note the different steps of the solution of a locus problem of this set.

[All that is necessary is (i) to assume (x, y) to be any point on the locus, (ii) to express the condition of the problem in terms of x, y and some known quantities, and (iii) to simplify the result of (ii), if necessary.]

CHAPTER II.

THE STRAIGHT LINE.

9.

ON STRAIGHT LINES AND EQUATIONS.

9·1. Every straight line can be represented by an equation of the first degree in x, y.

Let AB be a straight line; and let $A \equiv (x_1, y_1)$ and $B \equiv (x_2, y_2)$ be two points on it.[1] Then, if $P \equiv (x, y)$ be any point on AB, the area of the $\triangle PAB$ must be zero.

Hence, by Art. 6·3, we have

$$\begin{vmatrix} x, & y, & 1 \\ x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \end{vmatrix} = 0.$$

$$\therefore\ x(y_1 - y_2) + y(x_2 - x_1) + (x_1 y_2 - x_2 y_1) = 0, \quad \ldots \quad \ldots \quad (1)$$

an equation of the form $Ax + By + C = 0$.

As this is an equation of the first degree in x and y, it is clear that a straight line *can* be represented by an equation of the first degree in x, y.[2-3]

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[1] To be given *two* points on a straight line is equivalent to be given the straight line itself. Hence, a straight line being given, we can take two points on it and consider them as known.

[2] It must have been already clear to the beginner that by "*the equation of a straight line (or a curve)*" is meant an equation between the co-ordinates of any point P on it (including some known quantities).

The co-ordinates of any point on a straight line (or a curve) are sometimes called its *current co-ordinates*.

[3] The choice of the axes of reference does not change the degree of an equation in x, y [Art. 4·41]. So, we would have got an equation of the first degree in all cases.

9·2. Every equation of the first degree in x, y represents a straight line.

The most general equation of the first degree in x, y is

$$Ax+By+C=0, \quad \dots \quad \dots \quad \dots \quad (2)$$

where none of the constants A, B, C is *zero*.

Let $P\equiv(x_1, y_1)$, $Q\equiv(x_2, y_2)$ and $R\equiv(x_3, y_3)$ be any three point on the locus represented by (2).

Then, we must have

$$Ax_1+By_1+C=0, \quad \dots \quad \dots \quad \dots \quad (3)$$

$$Ax_2+By_2+C=0 \quad \dots \quad \dots \quad \dots \quad (4)$$

and $$Ax_3+By_3+C=0. \quad \dots \quad \dots \quad \dots \quad (5)$$

Eliminating A, B, C from (3), (4), (5),[4] we have

$$\begin{vmatrix} x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \\ x_3, & y_3, & 1 \end{vmatrix} = 0. \quad \dots \quad \dots \quad (6)$$

Hence, the area of the triangle formed by joining any three points on the locus is zero. Hence, (2) *must represent a straight line.*[5] And since any equation of the first

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[4] To eliminate A, B, C from these equations, we multiply the first by y_2-y_3, the second by y_3-y_1 and the third by y_1-y_2, and add together the results thus obtained. This gives

$A\{x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)\}=0.$

But $A\neq 0$, by hypothesis. Hence, we have $x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)=0$ as the *eliminant* (or the condition of coexistence) of (3), (4) and (5).

It can be written as $\begin{vmatrix} x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \\ x_3, & y_3, & 1 \end{vmatrix}=0$

as can be easily verified by expanding the determinant. (This is a very convenient form of the eliminant in such a case, and can be remembered very easily).

[5] Note that this part of the proof is based on the assumption that none of the constants A, B, C is zero. It *fails* when any one of these constants vanishes.

degree in x, y can be transformed into one of the type (2),[6] and the geometrical nature of the locus cannot be changed thereby,[7] it is clear that *every equation of first degree in x, y represents a straight line.*[8]

9·21. On the conditions necessary to determine a straight line.

The most general equation of a straight line, *viz.* $Ax+By+C=0$ appears to contain three constants. But since we can divide it by any constant without changing the nature of the relation indicated by it, the number of *independent* constants in the equation of a st. line is really two (the ratios of two of the quantities with respect to the third).

Now, two equations are both *necessary and sufficient* for the determination *of two* quantities. *Hence, a straight line can be made to satisfy two (and only two) conditions*, these conditions being such that each may give rise to one equation between the constants. Thus, a straight line can be made to pass through two given

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[6] Thus, by applying the most general formulae of transformation given in Art. 4·3, we can transform $x=0$ into $h+x_1\cos\theta-y_1\sin\theta=0$, an equation of the type (2); and so on.

[7] The whole thing becomes meaningless otherwise.

[8] Another proof of this result may be given as follows :—

Comparing (1) and (2), we have

$$\frac{A}{y_1-y_2}=\frac{B}{x_2-x_1}=\frac{C}{x_1y_2-x_2y_1}.$$

Now, as four quantities x_1, x_2, y_1, y_2 can be easily chosen to satisfy two equations, it is clear that (2) can be put in the form of (1). Hence, the area of the triangle formed by joining the points $P\equiv(x_1, y_1)$, $Q\equiv(x_2, y_2)$ and any point (x, y) on the locus is zero. Hence, the locus of (x, y) is the straight line PQ. Hence the result.

points, or to pass through a given point and be parallel to a given straight line.[9-11]

9·3. Every homogeneous equation of the second degree in *x*, *y* represents two straight lines (real or imagniary) through the origin.

Let $ax^2+2hxy+by^2=0$ be the most general homogeneous equation of the second degree in x and y, where none the constants a, h, b is *zero*.

Dividing it throughout by x^2, we have

$$b\left(\frac{y}{x}\right)^2+2h\left(\frac{y}{x}\right)+a=0. \qquad \ldots \qquad \ldots \qquad (7)$$

And if m_1, m_2 be the roots (real or imaginary) of this equation in $\frac{y}{x}$, then (7) is the same as

$$b\left(\frac{y}{x}-m_1\right)\left(\frac{y}{x}-m_2\right)=0. \qquad \ldots \qquad \ldots \qquad (8)$$

But $b\neq 0$, by hypothesis. Hence, (8) is satisfied when and only when at least one of the other factors is zero. Hence, the locus represented by (7) consists of the loci represented by the following equations :—

$$y-m_1x=0, \quad y-m_2x=0.$$

Now, as each of these equations is of the first degree in x and y, it represents a straight line. Moreover, it

[9] The conditions should be independent of each other as in the cases quoted here; otherwise, an *infinite* number of straight lines would satisfy them.

[10] Any condition which gives rise to two independent equations between the constants must, for this purpose, be regarded as equivalent to two independent conditions.

[11] The equations of condition may give rise to more than one set of values of these constants. In that case, more than one straight line will satisfy the given conditions, the number of straight lines satisfying two independent conditions being, of course, *finite*.

can be easily seen that the origin lies on each of them.[12] Hence, the equation $ax^2 + 2hxy + by^2 = 0$, where none of the constants a, h, b is *zero*, represents two straight lines (real or imaginary) through the origin.

And as *any* homogeneous equation of the second degree in x, y can be transformed into one of the most general type by a rotation of axes without change of origin, it is clear that every such equation represents two straight lines (real or imaginary) through the origin.[13]

9·4. To find the condition that an equation of the second degree in x, y may represent two straight lines (real or imaginary).

If the general equation of the second degree represents two straight lines, then by transforming the origin to the point of intersection of these lines, we shall get an equation which must be homogeneous and of the second degree in the new variables.

Now, the most general equation of the second degree in x, y is $ax^2 + 2hxy + by^2 + 2fy + 2gx + c = 0$, ... (9) where a, h, b, f, g and c are arbitrary constants (none being *zero*). If this represents two straight lines, then by changing the origin to their point of intersection (x', y') without changing the directions of the axes, we have

$$a(x_1 + x')^2 + 2h(x_1 + x')(y_1 + y') + b(y_1 + y')^2 + 2f(y_1 + y') + 2g(x_1 + x') + c = 0.$$ [14]

[12] The origin lies on the curve represented by an equation in which the absolute term (the term free from x and y) is absent.

[13] Similarly, it can be easily seen that a homogeneous equation of the third degree in x, y represents three straight lines (real or imaginary) through the origin; and so on.

[14] The formulæ of transformation in this case are $x = x_1 + x'$ and $y = y_1 + y'$ (Art. 4·1).

And as this must be homogeneous and of the second degree in x, y, the coefficients of x_1, y_1 and the absolute term must be zero. Hence, we have

$$ax' + hy' + g = 0, \quad \ldots \quad \ldots \quad \ldots \quad (10)$$

$$hx' + by' + f = 0 \quad \ldots \quad \ldots \quad \ldots \quad (11)$$

and $$ax'^2 + 2hx'y' + by'^2 + 2fy' + 2gx' + c = 0. \quad \ldots \quad (12)$$

But the left-hand side of $(12) \equiv x'(ax' + hy' + g) + y'(hx' + by' + f) + gx' + fy' + c$

$= gx' + fy' + c$, by (10) and (11).

Hence, the equations (10), (11) and (12) are together equivalent to the set consisting of (10), (11) and

$$gx' + fy' + c = 0. \quad (13)$$

Eliminating x', y' from (10), (11) and (13), we have

$$\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix} = 0 \quad \ldots \quad (14)$$

as a *necessary condition* for the general equation of the second degree to represent two straight lines (real or imaginary).[15–16]

9·41. Conversely, if (14) holds good, the general equation of the second degree as written above will represent two straight lines.

[15] It may be noted that *three equations are necessary and sufficient for eliminating two independent quantities from them*—two for determining the values of the two quantities, and one more in which to substitute the values of the quantities thus obtained in order to get at the *eliminant* (or the condition of co-existence).

[16] The quantities A, B, C of Art. 9·2 are really equivalent to two *independent* quantities, *viz*. the ratios of two of them with respect to the third. Hence, we could eliminate them from *three* equations. In fact, the number of independent equations should be one more than the number of independent variables to be eliminated.

If (14) holds good, we can find x', y' so that

$$ax' + hy' + g = 0, \quad \ldots \quad \ldots \quad (15)$$

$$hx' + by' + f = 0, \quad \ldots \quad \ldots \quad (16)$$

$$\text{and } gx' + fy' + c = 0, \quad \ldots \quad \ldots \quad (17)$$

since two quantities can always be chosen so that they may satisfy two equations in them, and the values of x', y' given by any two of these equations satisfy the third by virtue of (14).

Now, transferring the origin to (x', y') as defined by (15) and (16) without changing the directions of the axes, and noting that

$$ax'^2 + 2hx'y' + by'^2 + 2fy' + 2gx' + c \equiv x' \times (15) + y' \times (16) + (17) = 0,$$

we have $ax_1^2 + hx_1y_1 + by_1^2 = 0 \quad \ldots \quad \ldots \quad (18)$

as the new form of (9).

This being a homogeneous equation of the second degree in x_1, y_1, it represents two straight lines (real or imaginary) through the origin. Hence, the converse of Art. 9·4 is true. Hence, (14) gives a *sufficient* condition for (9) to represent two straight lines (real or imaginary).[17-19]

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[17] The *necessary and sufficient* condition that the most general equation of the second degree (in which none of the terms are missing) may represent two straight lines (real or imaginary) may also be obtained as follows :—

Multiplying (9) by a, and re-arranging, we have

$a^2x^2 + 2ax(hy + g) = -aby^2 - 2afy - ac.$

$\therefore \quad (ax + hy + g)^2 = y^2(h^2 - ab) + 2y(gh - af) + g^2 - ac,$

or $\quad ax + hy + g = \pm\sqrt{y^2(h^2 - ab) + 2y(gh - af) + g^2 - ac}.$

Now, in order that this may be reducible to a linear form, it is *necessary and sufficient* that the expression under the radical sign be a perfect square. The condition for this is

9·5. Polar equation of a straight line.

Changing $Ax+By+C=0$ to polar co-ordinates, we have $Ar\cos\theta+Br\sin\theta+C=0$.

And dividing throughout by $\sqrt{A^2+B^2}$, we have

$$\frac{A\ r\cos\theta}{\sqrt{A^2-B^2}}+\frac{B\ r\sin\theta}{\sqrt{A^2+B^2}}+\frac{C}{\sqrt{A^2+B^2}}=0. \quad \ldots (19)$$

Now choose α so that $\dfrac{\cos\alpha}{A}=\dfrac{\sin\alpha}{B}=\dfrac{1}{\sqrt{A^2+B^2}}$.

Then, (19) becomes

$$r(\cos\theta\cos\alpha+\sin\theta\sin\alpha)=p, \text{ where } p=\frac{-C}{\sqrt{A^2+B^2}}.$$

$$\therefore\ r\cos(\theta-\alpha)=p, \quad \ldots \quad \ldots \quad \ldots \quad (20)$$

which may be regarded as the *standard form* of the equation of a straight line in polar co-ordinates.

*9·6. The position in oblique co-ordinates.

As the *degree* of an equation is unaltered by any transformation of Cartesian axes, it is clear that the equation $Ax+By+C=0$ represents a straight line even when the axes are *oblique*.

$$(h^2-ab)(g^2-ac)=(hg-af)^2,$$

$$\text{or}\quad a\{abc+2fgh-af^2-fg^2-ch^2\}=0.$$

But $a\neq0$. Hence, $abc+2fgh-af^2-bg^2-ch^2=0$, which is the same as (14).

[18] The method given in foot-note 17 is quite simple, though it requires a careful beginning when some of the terms are absent. But the method given in the body of the book is very good in this respect. It will, moreover, be found to be very useful in the discussion of some of the allied topics—finding the co-ordinates of the point of intersection of the straight lines represented by (9), etc.

[19] Another method (and one that can be applied to equations of higher degrees) of finding the condition that (9) may represent two straight lines is to equate the left-hand side of (9) to the product of two linear factors, write down the conditions under which this is possible, and then eliminate the unknown quantities from them. The *sufficiency* of the condition thus found can be proved by proceeding backward from (14).

Again, since the proof of the result of Art. 9·3 does not require the property (or any deduction thereof) that distinguishes rectangular axes from oblique axes, it is clear that the result of that article is true even when the axes are *oblique*. Similarly, (14) gives the condition that must be satisfied in order that the most general equation of the second degree in oblique co-ordinates may represent two straight lines (real or imaginary).[20]

Examples.

1. Show that $ax+by+c+\lambda(a'x+b'y+c')=0$ represents a straight line passing through the point of intersection of the straight lines $ax+by+c=0$ and $a'x+b'y+c'=0$.

[*Solution.* As the equation is of the first degree in x, y, it represents a straight line; and since it is satisfied whenever $ax+by+c=0$ and $a'x+b'y+c'=0$ are simultaneously satisfied, it is clear that the first straight line passes through the intersection of the last two.]

2. Prove that the straight line $2x(\gamma+1)+3y(\gamma-2)+4-3\gamma=0$ passes through a fixed point for all values of γ.

[*Solution.* (*i*) *Re-arranging*, we have

$$2x-6y+4+\gamma(2x+3y-3)=0.$$

(*ii*) *This shows that it passes through the point of intersection of the straight lines* $x-3y+2=0$ and $2x+3y-3=0$;

[20] As the discussion, in most cases, has been *purposely* restricted to rectangular axes, it should *not* be taken for granted that the results are necessarily true when (and only when) the axes are rectangul

(*iii*) And as these straight lines are fixed, their point of intersection is also fixed. Hence the result.] [21]

3. Find the co-ordinates of the point of intersection of the straight lines $3x+4y+5=0$ and $6x+7y+1=0$.

[*Hints.* The point of intersection is on both of them. So, its co-ordinates are obtained by solving them as simultaneous equations in x, y.]

4. *Find the co-ordinates of the point of intersection of two straight lines whose equations are given.*

5. Find the co-ordinates of the fixed point through which the straight line of Ex. 2 passes.

6. *Find the condition that three lines whose equations are given may be concurrent.*

[*Hints.* The three equations hold together at the point of concurrency, and at no other points. So, the condition of concurrency is obtained by eliminating x and y from the given equations.] [22]

7. Show that the straight lines $x+y-3=0$, $2x-3y+5=0$ and $5x-5y+7=0$ are concurrent.

8. Show that the straight lines $\delta_1=\delta_2=\delta_3=0$ will meet in a point, if three constants λ_1, λ_2, λ_3 can be found so that $\lambda_1\delta_1+\lambda_2\delta_2+\lambda_3\delta_3$ may vanish *identically*. Hence show that the straight lines of Ex. 7 are concurrent.

---

[21] If we had rearranged the equation as $(2x+3\gamma y+4)+(2\gamma x-6y-3\gamma)=0$, we could say that the straight line passes through the point of intersection of $2x+3\gamma y+4=0$ and $2\gamma x-6y-3\gamma=0$. But as the point of intersection is not a fixed point in this case, such a procedure (though *logically* correct) would not lead us to the result in view.

[22] A knowledge of the use of determinants in the process of elimination helps us in writing down the condition at once—without going through the fundamental processes (of solution and substitution)

9. Find λ so that the equation

$$3x^2 + \lambda xy + 4y^2 + 5x - 7y + 9 = 0$$

may represent a pair of straight lines.

**10. *Assuming that the general equation of the second degree represents two straight lines, find their point of intersection, and write down their equations separately.*

[*Solution.* From Art. 9·4, it is clear that the point of intersection (x', y') is given by $ax' + hy' + g = 0$ and $hx' + by' + f = 0$. Solving these equations for x' and y', we have $x' = \dfrac{hf - bg}{ab - h^2}$ and $y' = \dfrac{gh - af}{ab - h^2}$ at the point of intersection.

For the second part, we know that the equation reduces to $ax_1^2 + 2hx_1y_1 + by_1^2 = 0$ by changing the origin to the point of intersection of the given lines without changing the directions of the axes. Hence, the straight lines are given by $\dfrac{x_1}{y_1} = -\dfrac{h \pm \sqrt{h^2 - ab}}{a}$ with reference to the new axes. Transforming back to the old axes, we have $\dfrac{x - x'}{y - y'} = -\dfrac{h \pm \sqrt{h^2 - ab}}{a}$ as the equations of the given straight lines (x', y' being already known).]

11. If $3x^2 + 2xy - y^2 - 7x + 5y + \lambda = 0$ represents two straight lines, what are their equations taken separately ?

12. *Find the conditions under which the general equation of the second degree represents two parallel straight lines.*

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on which the conclusion is based. For further information on the various methods of elimination, see Hall and Knight's *Algebra*, or Burnside and Panton's *Theory of Equations*, Vol. II.

**13. *Show that every homogeneous equation of the nth degree in x, y represents n straight lines (real or imaginary) through the origin.* [23]

[*Solution.* Let $ay^n + hy^{n-1}x + cy^{n-2}x^2 + \dots + kx^n = 0$ ... ... (21)

be the most general equation of the nth degree in x, y where $a, h, c, \dots \dots \dots k$ are arbitrary constants (none being *zero*). Dividing (21) by x^n throughout, we have

$$a\left(\frac{y}{x}\right)^n + b\left(\frac{y}{x}\right)^{n-1} + \dots \dots + k = 0. \quad \dots \quad (22)$$

Assuming that a rational integral equation of the nth degree in a single variable possesses n roots (real or imaginary), and supposing that $m_1, m_2, m_3, \dots\dots m_n$ are the roots of (22), we can write (22) in the form

$$a\left(\frac{y}{x} - m_1\right)\left(\frac{y}{x} - m_2\right)\left(\frac{y}{x} - m_3\right)\left(\frac{y}{x} - m_n\right) = 0. \quad \dots \quad (23)$$

But $a \neq 0$, by hypothesis. Hence, (23) is satisfied when, and only when, at least one of the other factors is zero. Hence, the locus represented by (21) consists of the loci represented by the following equations :—

$$y - m_1 x = 0, \quad y - m_2 x = 0, \quad y - m_3 x = 0 \text{ and } y - m_n x = 0.$$

Now, as these equations are all of the first degree in x, y, they represent so many straight lines. Moreover, the origin lies on each of them. Hence, (21) represents n straight lines (real or imaginary) through the origin.

And as any homogeneous equation of the nth degree in x, y can be transformed into one of the most general type,

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[23] Proceeding as in Art. 9·3, and assuming that an equation of this type possess n roots (real or imaginary), we should be able to prove this theorem. Details are, however, given here for the convenience of beginners.

it is clear that every such equation represents n straight lines (real or imaginary) through the origin.

** 14. How many conditions must be satisfied in order that the most general equation of the third degree in x, y, may represent three concurrent straight lines ?

[*Solution.* If it represents three concurrent straight lines, then by transforming the origin to that point without changing the directions of the axes, we must get an equation which must be homogeneous and of the third degree in the new variables. So, the terms of the first and second degrees and the absolute term must vanish separately. This will give six equations between two quantities (the co-ordinates of the point of concurrency). So, there must be $(6-2)$ or 4 conditions between the nine independent constants occurring in the most general equation of the third degree in x, y.

** 15. Find the number of conditions that must be satisfied in order that the most general equation of the fourth degree may represent four straight lines.

10.

GEOMETRICAL INTERPRETATIONS OF CONSTANTS IN THE EQUATION OF A STRAIGHT LINE.

For this purpose, we shall write the equation of a straight line in three different forms (known as *standard forms*), viz.

(i) $\frac{x}{a} + \frac{y}{b} = 1$, (the intercept form),

(ii) $y = mx + c$, (the slope form),

and (iii) $x \cos \alpha + y \sin \alpha - p = 0$,
p being the positive (the normal form).

(i) Consideration of the intercept form. [24]

If the equation of a straight line is written in the form $\frac{x}{a}+\frac{y}{b}=1$, then it is easy to see that $x=a$, when $y=0$ and $y=b$, when $x=0$ (fig. 18). Hence, in this case, a and b are the intercepts (in magnitude and sign) on the axes of x and y respectively.

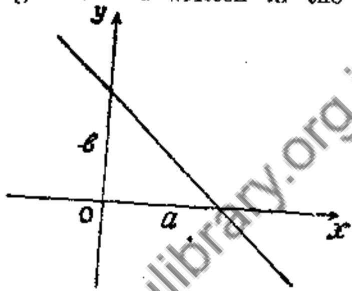

FIG. 18

(ii) Consideration of the slope form.

Putting $x=0$ in $y=mx+c$, we have $y=c$. Hence, c *gives the intercept on the y-axis (in magnitude and sign).*

Again, putting $y=0$ in the same equation, we have $x=-\frac{c}{m}$. Now, representing the state of affairs by means of a diagram (fig. 19), we see that

$$\tan xAB=\frac{OB}{AO}=\frac{c}{\frac{c}{m}}=m,$$

where the angle xAB is the angle which the straight line AB makes with the axis of x (fig. 19). [25]

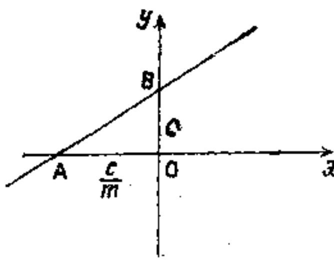

FIG. 19

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[24] By *the intercept on the axis of x* is meant the length of x-axis intercepted between the origin and the point where the given straight line cuts that axis.

[25] In representing the state of affairs as in fig. 19, we supposed that c and $\frac{c}{m}$ (and hence both c and m) are *positive*. That the final

Hence, m *may be regarded as the tangent of the angle which the upward direction of the straight line makes with the axis of* x.[26]

(iii) Consideration of the normal form.

Putting $y=0$ and $x=0$ successively in $x\cos\alpha+y\sin\alpha-p=0$ (p being positive), we see that the intercepts on the axes of x and y are $\frac{p}{\cos\alpha}$ and $\frac{p}{\sin\alpha}$. Now, representing the state of affairs by means of a diagram (fig. 20), and drawing $ON\perp$ to the given straight line AB, we have

$$ON = OA\cos xON$$
$$= OB\sin xON$$

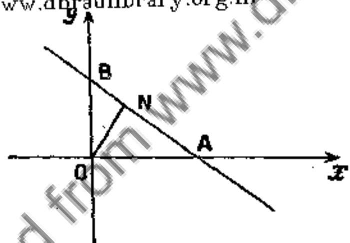

FIG. 20

$$\therefore \frac{p}{\cos\alpha}\cdot\cos xON = \frac{p}{\sin\alpha}\cdot\sin xON$$

$$\text{or}\quad \tan\alpha = \tan xON.$$

Now, the angle xON is the vectorial angle of the foot of the perpendicular from the origin to the given

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result *does not* depend on these assumptions can be easily seen by proceeding independently with a different diagram of the state of affairs, and noting the signs which the quantities must have in that case.

[26] In what follows, we shall suppose m to have this meaning. When measured positively, the magnitude of this angle may be taken to be less than two right angles.

straight line with reference to the x-axis as the initial line, and the origin as the pole. *Hence, α may be regarded as the vectorial angle of the foot of the perpendicular from the origin to the given straight line.*

Again, taking $\alpha \equiv \angle xON$ (as will be done in what follows), we have

$$ON = OA \cos xON = \frac{p}{\cos \alpha} \cdot \cos xON = p.$$

Hence, p *is equal to the length of the perpendicular from the origin to the given straight line.* [27]

Hence, we see that the constants in the equation of a straight line as given by (i), (ii) or (iii) admit of simple geometrical interpretations.

10·01. Note on the various forms in which the equation of a straight line has been written.

As the equation of a straight line must always be of the first degree in x, y, it is clear that every form of its equation can be reduced to every other form (although a suitable choice of the form of the equation may simplify matters in many geometrical problems). Thus :—

(*i*) To reduce $Ax + By + C = 0$ (where C is positive) to the normal form $x \cos \alpha + y \sin \alpha - p = 0$, we first write the equation as $-Ax - By - C = 0$ with a negative sign prefixed to C, and then choose p and α so that the last two equations may be the same. Now, comparing the two equations, we have $\dfrac{\cos \alpha}{-A} = \dfrac{\sin \alpha}{-B} = \dfrac{p}{C} = k$ (say).

[27] Note that the polar co-ordinates of the foot of the perpendicular from the origin to the straight line $x \cos \alpha + y \sin \alpha - p = 0$ (p being positive) are (p, α)—with reference to the origin as the pole, and the x-axis as the initial line.

$$\therefore \quad \cos^2 \alpha + \sin^2 \alpha = k^2(A^2 + B^2).$$

$\therefore\ k = \dfrac{1}{\sqrt{A^2 + B^2}}$, taking the radical with positive sign.[28]

$$\therefore \quad \cos \alpha = \frac{-A}{\sqrt{A^2 + B^2}}, \quad \sin \alpha = \frac{-B}{\sqrt{A^2 + B^2}}$$

$$\text{and } \ p = \frac{C}{\sqrt{A^2 + B^2}}.$$

Hence, the given equation in its normal form is

$$-\frac{Ax}{\sqrt{A^2 + B^2}} - \frac{By}{\sqrt{A^2 + B^2}} - \frac{C}{\sqrt{A^2 + B^2}} = 0 \quad \ldots \quad (2)$$

(it being possible now to regard the co-efficients of (2) as those in the normal form).[29–30]

(*ii*) Again, to reduce $Ax + By + C = 0$ to the intercept form $\dfrac{x}{a} + \dfrac{y}{b} = 1$, we must find a and b so that the two equations may be the same. Now, comparing the equations,

we have $$\frac{A}{\frac{1}{a}} = \frac{B}{\frac{1}{b}} = \frac{-C}{1}.$$

$\therefore\ a = \dfrac{-C}{A}$, $b = \dfrac{-C}{B}$. Hence, the intercept form of

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[28] Since both p and C are positive, the value of the ratio k *cannot* be negative.

[29] Thus, the *mechanical rule* for reducing the equation $Ax+By+C=0$ (where C is positive) to its normal form is (*i*) to write it as $-Ax-By-C=0$, and (*ii*) to divide it throughout by $+\sqrt{A^2+B^2}$.

[30] By reducing the equation of a straight line to its normal form, we can easily find the length of the perpendicular from the origin to the given straight line as well as the angle which this perpendicular makes with the axis of x.

the given equation is

$$\frac{x}{\frac{-C}{A}}+\frac{y}{\frac{-C}{B}}=1.$$

Similarly, any other case can be discussed.

*10·1. The position in oblique co-ordinates.

In the intercept form $\frac{x}{a}+\frac{y}{b}=1$, a and b have the same meanings as when the axes are *rectangular*.

In $y=mx+c$, c represents the intercept on the y-axis even when the axes are *oblique*; but m has not the same meaning as before. Representing the state of affairs by means of a diagram, it can be easily seen that $m=\frac{\sin\beta}{\sin(\omega-\beta)}$, where β is the angle which the upward direction of the given straight line makes with the x-axis.

The equation corresponding to the normal form is $x\cos\alpha+y\cos\beta-p=0$, where $\beta=\omega-\alpha$.

Examples.

1. Reduce the following equations to their normal forms :—

$3x-2y+4=0$, $\frac{x}{3}-\frac{y}{4}=1$ and $y=3x+\frac{1}{2}$.

2. Reduce $\frac{x}{a}+\frac{y}{b}=1$ to the slope form.

3. Find the angle which the straight line $3x-4y+2=0$ (*i.e.*, its upward direction) makes with the axis of x.

[*Solution.* (i) Reducing the given equation to the form $y = mx + c$, we have $y = \frac{3}{4}x + \frac{1}{2}$. Hence, the angle is $\tan^{-1}\frac{3}{4}$.

(ii) Now, since the angle (taken positively) lies between 0 and π, and its tangent is positive in this case, it is that acute angle whose tangent is $\frac{3}{4}$.]

4. Find the obtuse angle between the straight line $x - y + 1 = 0$ and the axis of x.

5. Find the lengths of the perpendiculars from the origin to the straight lines of Ex. 1.

6. Find the acute angle between the line which is perpendicular from the origin to the straight line $3x - 4y + 2 = 0$ and the axis of x.

[*Hints.* Put the equation in its normal form to find the vectorial angle of the foot of the perpendicular. The required acute angle can then be easily found.]

7. Find the Cartesian and polar co-ordinates of the foot of the perpendicular from the origin to the straight line $3x - 4y + 2 = 0$.

8. Find the intercepts which the straight lines of Ex. 1 make on the axes.

11.

ON THE RELATIVE POSITIONS OF POINTS AND STRAIGHT LINES.

A point is said to lie on a straight line (or a curve) if its co-ordinates satisfy the equation of the straight line (or the curve). We shall exclude this case from our consideration just now.

11·1. On the condition that two points may lie on the same or opposite sides of a given st. line.

Let $Ax+By+C=0$ be the st. line; and $P\equiv(x_1, y_1)$, $Q\equiv(x_2, y_2)$ be the given points. Then, the co-ordinates of the point R which divide in the ratio $m_1 : m_2$ the line joining P and Q are $\dfrac{m_1x_2+m_2x_1}{m_1+m_2}$ and $\dfrac{m_1y_2+m_2y_1}{m_1+m_2}$.

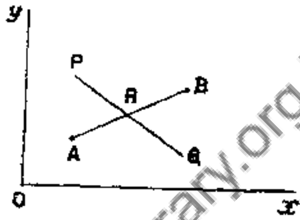

FIG. 21

If this point be on the given line AB, we must have

$$A\,\frac{m_1x_2+m_2x_1}{m_1+m_2}+B\,\frac{m_1y_2+m_2y_1}{m_1+m_2}+C=0.$$

$$\therefore\ \frac{m_1}{m_2}=-\frac{Ax_1+By_1+C}{Ax_2+By_2+C}.\quad \ldots\quad \ldots\quad \ldots\quad (1)$$

Now, the ratio $\dfrac{m_1}{m_2}$ is *negative* or *positive* according as R lies *without* or *within* the finite segment PQ,[31] *i.e.*, according P and Q are on the same or opposite sides of the given line AB. But from (1), it is clear that the ratio $\dfrac{m_1}{m_2}$ is *negative* or *positive* according as Ax_1+By_1+C and Ax_2+By_2+C have the same or opposite signs. Hence, the points (x_1, y_1) and (x_2, y_2) lie on the same or opposite sides of $Ax+By+C=0$ according as Ax_1+By_1+C and Ax_2+By_2+C have the same or opposite signs.[32–33]

[31] PR and RQ having the same or opposite signs according as R is within or without the *finite* segment PQ.

[32] Hence, *when a point $P\equiv(x_1, y_1)$ passes from one side of $Ax+By+C=0$ to another, the expression Ax_1+By_1+C changes sign.*

[33] Hence, the *mechanical rule* for finding whether two points lie on the same or opposite sides of $Ax+By+C=0$ is (i) to *substitute* their co-ordinates in the expression $Ax+By+C$ and (ii) to note whether the signs of the resulting expressions are the same or not.

11·11. Notes on the above.

In the special case when one of the points is the origin, a point (x_1, y_1) will be on the same side of the given straight line as the origin or not according as $Ax_1 + By_1 + C$ has the same sign as C or not.

When it is desired to find whether a given point is within or without that angle between two given lines in which the origin lies, we should write the given equations with the absolute terms of both having the same sign before proceeding further.

Examples.

1. Show that the points (5, 6) and (2, −1) lie on the opposite sides of the line $3x + 4y - 7 = 0$

2. Is the origin on the same side of $x + y = 5$ as (3, 4)? Is the middle point of the line joining the origin to (3, 4) on the same side of the given line as the origin?

3. Find whether the point (3, 4) lies within or without that angle between the straight lines $3x + 4y + 7 = 0$ and $x + y - 5 = 0$ in which the origin lies.

[*Solution.* Write the first equation as $-3x - 4y - 7 = 0$ so that absolute terms in both may be *negative*. The co-ordinates of the origin will now make the expressions on the left-hand side of each equation negative. Again, the point (3, 4) makes the expression $-3x - 4y - 7$ negative, and the expression $x + y - 5$ positive. Hence, the point (3, 4) lies on the same side of the line $-3x - 4y - 7 = 0$ as the origin; but this is not true in the second case. Now, representing the state of affairs by means of a diagram, it can be easily seen that the point cannot lie within the specified angle in such a case. Hence, we conclude that the point lies *without* that angle.]

4. Find whether the point (1, 2) lies within or without that angle between the straight lines of Ex. 3 in which the origin lies.

5. Show that the point (3, 4) lies within that angle between the straight lines $3x-5y-9=0$ and $2x+y-15=0$ in which the origin lies.

12.

ON THE PERPENDICULAR DISTANCE OF A GIVEN POINT FROM A GIVEN STRAIGHT LINE.

12·1. When the equation of a straight line is given in its normal form $x \cos\alpha + y \sin \alpha - p = 0$ (p being positive).

Let $P \equiv (x', y')$ be the given point. Now, the equation of the given straight line with reference to parallel axes through the point P is

$$x_1 \cos\alpha + y_1 \sin \alpha - p_1 = 0, \quad \ldots \quad \ldots \quad (1)$$

where $p_1 \equiv p - x' \cos \alpha - y' \sin \alpha$ and x_1, y_1 are the new current co-ordinates of the given line.

Comparing this with the normal form of the equation of a straight line, and *noting* that p_1 is *not necessarily positive*, we conclude that the absolute value of p_1 gives the magnitude of the distance of P from the given straight line.[34]

[34] If p_1 is negative, the normal form of the equation (1) is $-x_1 \cos \alpha - y_1 \sin \alpha - (-p_1) = 0$, or $x_1 \cos(\alpha+\pi) + y_1 \sin(\alpha+\pi) - (-p_1) = 0$.

12·2. When the equation of the straight line is given in any other form.

In any other case, the perpendicular distance of any point $P \equiv (x', y')$ from a given straight line AB can be found (i) by *reducing* the equation of AB to its normal form $x \cos \alpha + y \sin \alpha - p = 0$, and (ii) taking the *absolute value* of the expression $p - x' \cos \alpha - y' \sin \alpha$.

12·3. Note on the sign of the expression $p - x' \cos \alpha - y' \sin \alpha$.

The expression $x \cos \alpha + y \sin \alpha - p$ becomes *negative* when the co-ordinates of the origin are substituted in it. Hence, $x' \cos \alpha + y' \sin \alpha - p$ will be *negative*, if $P \equiv (x', y')$ lies on the same side of the given line $x \cos \alpha + y \sin \alpha - p = 0$ as the origin. Hence, $p - x' \cos \alpha - y' \sin \alpha$ will be *positive* or *negative* according as $P \equiv (x', y')$ lies on the same side of the given line as the *origin* or on the opposite side of it.[35] Hence, the length of the perpendicular changes sign (as it should do) with the change of the position of (x', y') from one side of $x \cos \alpha + y \sin \alpha - p = 0$ to the other.

If the equation of the straight line is given in any other form, we can easily discuss this question by reducing it to its normal form.

Examples.

1. Find the lengths of the perpendiculars from $(2, -3)$ on the following straight lines :—

(i) $3x - 2y + 4 = 0$, (ii) $y = 2x + 3$ and (iii) $\dfrac{x}{5} - \dfrac{y}{4} = 4$.

---

[35] Hence, if we know the positions of P and AB, we know the sign of $p - x' \cos \alpha - y' \sin \alpha$ with the help of the above criterion.

*2. Find the length of the perpendicular from the origin to $y=mx+c$, the axes being *oblique*.

[*Solution.* If β be the angle the straight line makes with the x-axis, we have $m=\dfrac{\sin\beta}{\sin(\omega-\beta)}$. And if ON be drawn perpendicular to it, we have $\dfrac{\sin\beta}{ON}=\dfrac{\sin 90^\circ}{c/m}$.

Hence, $ON=c/m.\sin\beta=\pm\dfrac{c\sin\omega}{\sqrt{1+m^2+2m\cos\omega}}$.]

3. Show that the product of the perpendiculars from the point (3, 4) on the straight lines given by $x^2+6xy-4y^2=0$ is equal to $\dfrac{17}{\sqrt{61}}$.

13.

ON THE ANGLE BETWEEN TWO STRAIGHT LINES WHOSE EQUATIONS ARE GIVEN.

13·1. When the equations of the straight lines are given in their slope forms.

Let the equations of the straight lines be $y=mx+c$ and $y=m'x+c'$; and let θ, θ' be the angles which these lines (*i.e*, their upward directions) make with the axis of x. Then, we have $\tan\theta=m$ and $\tan\theta'=m'$. Now, the angle between them is $\theta\sim\theta'$.

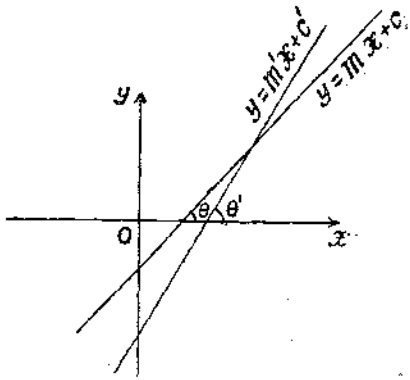

FIG. 22

Denoting $\theta - \theta'$ by ϕ, we have

$$\tan \phi = \tan (\theta - \theta').$$

$$= \frac{\tan \theta - \tan \theta'}{1 + \tan\theta \tan \theta'} = \frac{m - m'}{1 + mm'}.$$

$$\text{Hence, } \phi = \tan^{-1} \frac{m - m'}{1 + m m'}. \quad \dots \quad \dots \quad (1)$$

[If $\tan^{-1} \dfrac{m - m'}{1 + m m'}$ be *positive*, it will give the *acute angle* between the lines; and if it be *negative*, it will give the *obtuse angle* between them.]

13·2. When the equations of the straight lines are given in any other manner.

In any other case, the angle can be found by (*i*) reducing the equations of the straight lines to their slope forms, and (*ii*) proceeding as in Art. 13·1. Thus :—

(*i*) *To find the angle between the lines* $ax + by + c = 0$ *and* $a'x + b'y + c' = 0$, we write them as

$$y = -\frac{a}{b}x - \frac{c}{b} \text{ and } y = -\frac{a'}{b'}x - \frac{c'}{b'}.$$

$\therefore \; m = -\dfrac{a}{b}$ and $m' = -\dfrac{a'}{b'}$. Hence, by (1), we have

$$\tan \phi = \frac{-\dfrac{a}{b} + \dfrac{a'}{b'}}{1 + \dfrac{a}{b}\dfrac{a'}{b'}} = \frac{a'b - ab'}{aa' + bb'}. \quad \dots \quad \dots \quad \dots \quad (2)$$

(*ii*) Again, *to find the angle between the lines given by the single equation* $ax^2 + by^2 + 2hxy = 0$, we proceed as if the equations of the straight lines were given separately in their slope forms. Thus :—

Let the equations of the given straight lines (taken separately) be $y-mx=0$ and $y-m'x=0$. Then, we must have $ax^2+2hxy+by^2 \equiv b(y-mx)(y-m'x)$.

$$\therefore \quad m+m' = -\frac{2h}{b} \text{ and } mm' = \frac{a}{b}.$$

$$\therefore \quad \tan\phi = \frac{m-m'}{1+mm'} = \pm\frac{\sqrt{(m+m')^2-4mm'}}{1+mm'}.$$

$$= \pm\frac{2\sqrt{h^2-ab}}{a+b}. \quad \ldots \ldots \quad (3)$$

13·3. Note on some special cases.

(*i*) In the special case when the straight lines are *perpendicular* to each other, we have $\tan\phi = \infty$. Hence, the condition of *perpendicularity* in any case is found by equating to zero the denominator of the expression for $\tan\phi$ in that case. Thus, $m+m'=0$ is the condition that $y=mx+c$ and $y=m'x+c'$ may be *perpendicular* to each other ; and so on.

(*ii*) In the special case when the straight lines are *parallel* to each other, we have $\tan\phi = 0$. Hence, the condition of parallelism in any case is found by equating to zero the numerator of the expression for $\tan\phi$ in that case. Thus, $m=m'$ is the condition that the straight lines of Art. 13·1 may be *parallel* ; and so on.

[If, however, it is known that two straight lines have a common point of intersection, then the condition of *parallelism* would be equivalent to the condition of *coincidence*.]

* 13·4. The position in oblique co-ordinates.

Let $y=mx+c$, $y=m'x+c'$ be two given straight lines. Then, $m=\dfrac{\sin\beta}{\sin(w-\beta)}$ and $m'=\dfrac{\sin\beta'}{\sin(w-\beta')}$, where β and β'

are the angles the straight lines make with the axis of x and ω = the angle between the co-ordinate axes. Now, the angle between the lines $=\beta-\beta'$; and

$$\tan\beta = \frac{m\sin\omega}{1+m\cos\omega} \text{ and } \tan\beta' = \frac{m'\sin\omega}{1+m'\cos\omega}.$$

$$\therefore \tan(\beta-\beta') = \frac{\tan\beta-\tan\beta'}{1+\tan\beta\tan\beta'}$$

$$= \frac{(m-m')\sin\omega}{1+mm'+(m+m')\cos\omega}.$$

This gives the tangent of the angle between the lines in terms of known quantities.

[In any other case, the angle can be found by reducing the problem to one of the type given above.]

Examples.

1. Find the angle between the straight lines $x-y+1=0$ and $2x+3y+2=0$.

2. Find the angle between the straight lines

$$x\cos\alpha+y\sin\alpha-p=0 \text{ and } \frac{x}{a}+\frac{y}{b}=1.$$

3. If $2x^2+3xy+y^2+2x-3y+\lambda=0$ represents two straight lines, find the angle between them.

[*Hints.* If the equation represents two straight lines (which it does for a special value of λ), it can be reduced to $2x_1^2+3x_1y_1+y_1^2=0$ by changing the origin to their point of intersection without changing the directions of the axes (Art. 9·41). Hence, by (3), the angle is $\tan^{-1}\frac{1}{3}$.]

** 4. If the general equation of the second degree represents two straight lines; find the angle between them.

** 5. Find the condition that one of the lines $ax^2+2hxy+by^2=0$ may be perpendicular to one of the lines $a'x^2+2h'xy+b'y^2=0$.

[*Hints.* Let $ax^2+2hxy+by^2 \equiv b(y-m_1x)(y-m_2x)$, and $a'x^2+2h'xy+b'y^2 \equiv b'(y-m_3x)(y-m_4x)$.

$$\left.\begin{aligned} \text{Then, } m_1+m_2 &= \frac{-2h}{b},\ m_3+m_4=\frac{-2h'}{b'} \\ m_1m_2 &= \frac{a}{b} \text{ and } m_3m_4=\frac{a'}{b'}. \end{aligned}\right\} \quad (4)$$

Now, if $y-m_1x=0$ is $\perp$ to $y-m_3x=0$, we must have

$$m_3m_1=-1. \quad \ldots \quad (5)$$

Hence, the required condition is obtained by eliminating m_1, m_2, m_3 and m_4 from the equations (4) and (5).] [36]

* 6. Find the conditon that one of the lines $ax^2+2hxy+by^2=0$ may be the same as one of the lines $a'x^2+2h'xy+b'y^2=0$

**7. Show that the pair of straight lines $ax^2+2hxy+by^2=0$ are at right angles to the pair $bx^2-2hxy+ay^2=0$.

* 8. Find the angle between the lines $ax^2+2hxy+by^2=0$, the axes being *oblique*.

---

[36] As the relations (4) cannot be changed by interchanging m_1 and m_2, m_3 and m_4 (jointly or severally), it is clear that the final result does not depend on the manner in which relation (5) is obtained.

14.

ON THE CONSTRUCTION OF EQUATIONS OF STRAIGHT LINES SATISFYING ASSIGNED CONDITIONS.

14·1. The direct method.

One of the methods (probably the most straight-forward one, though not necessarily the simplest one) of constructing the equation of a stiaight line which satisfies given conditions is (i) to assume the equation of the required straight line in a suitable manner, (ii) to express the given conditions analytically, and (iii) to substitute in (i) the values of the constants as found from the equations of condition obtained in (ii). It is easily applicable to problems of the following types :—

(I) *To find the equation of the straight line which passes through* $(2, 3)$, *and is parallel to* $3x-2y=1$.

(*i*) Let the equation of the straight line be $y=mx+c$.

(*ii*) If it passes through $(2, 3)$, we must have
$3=2m+c$. ... ... (1)

Again, if it is parallel to $3x-2y=1$, then the m's of the two straight lines must be the same.

$\therefore \quad m=\frac{3}{2}$. ... ... (2)

(*iii*) Solving (1) and (2) for the constants m and c, we have $m=\frac{3}{2}$, $c=0$. And substituting their values in (1), we have $y=\frac{3}{2}x$ as the equation of the required straight line.

(II) *To find the equation of the straight line which passes through* $P\equiv(x_1, y_1)$ *and* $Q\equiv(x_2, y_2)$.

Let the equation of the straight line be $y = mx + c$.

If it passes through P and Q, we must have

$$y_1 = mx_1 + c, \text{ and } y_2 = mx_2 + c.$$

Eliminating m and c from these three equations, we have

$$\begin{vmatrix} y, & x, & 1 \\ y_1, & x_1, & 1 \\ y_2, & x_2, & 1 \end{vmatrix} = 0$$

as the equation of the straight line [*cf.* Art. 9·1].[37-38]

14·2. The method of undetermined multipliers.[39]

By making a suitable use of the fact that $S + \lambda S' = 0$ (where λ is an undetermined multiplier) represents a locus through the points of intersection of $S = 0$ and $S' = 0$, we can simplify matters to some extent in problems of the following types :—

(1) *To find the equation of the straight line which passes through the intersection of the lines* $3x - 2y = 1$ *and* $5x + y - 3 = 0$, *and is perpendicular to* $y = 4x$.

---

[37] The process of *elimination* may be regarded as equivalent to the processes of *solution and substitution* (jointly). But, since a previous knowledge of things helped us in writing down the final result at once, we did not go through the working details in this case.

[38] The equation of this straight line can also be written in the following *symmetrical* form :—

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}.$$

[39] It is not that problems solvable by this method cannot be solved by the previous method (or any other method). The whole thing depends on the nature of the start made in solving a problem, and on our ability to overcome the analytical difficulties. *The chief object of this classification* (based on experience) *is to help the beginner in analysing his thoughts in a more or less systematical manner, and to make him feel that there are not as many methods as*

(*i*) The equation of any straight line passing through the intersection of the given lines is $3x-2y-1+\lambda(5x+y-3)=0$, where λ is an undetermined multiplier.[40]

(*ii*) If this is perpendicular to $y=4x$, the product of the m's of the two lines must be $=-1$.

$$\therefore \quad 4\left(-\frac{3+5\lambda}{\lambda-2}\right)=-1, \text{ or } \lambda=-\frac{14}{19}.$$

(*iii*) Substituting this value of λ in (i), we have $13x+52y-23=0$ as the equation of the required straight line.[41]

* * (II) *To find the equations of the diagonals of the quadrilateral the equations of whose sides (taken in order) are $y=2x+3$, $y=5x$, $x+y+2=0$ and $x+1=0$.*

One of the diagonals is the line joining the point of intersection of the first two lines with that of the last two.

(*i*) Regarding it as a line through the intersection of the first two lines, we can write it as

$$y-2x-3+\lambda(y-5x)=0. \qquad \ldots \qquad \ldots \qquad (4)$$

(*ii*) Again, regarding it as a line through the intersection of the last two lines, we can write it as

$$x+y+2-\mu(x+1)=0. \qquad \ldots \qquad \ldots \qquad (5)$$

---

there are problems. It is hoped that the beginner will get a mastery over this important subject by noting the nature of the start made in each of the problems solved to illustrate the so-called methods, and by making similar starts in allied cases.

[40] Being an equation of the first degree in x, y, it represents a straight line; and since it is satisfied whenever $3x-2y-1=0$ and $5x+y-3=0$ are simultaneously satisfied, it passes through the point of intersection of the given lines.

[41] Compare the different steps of this method with those of *the direct method* (preferably after solving these problems by the latter method).

(*iii*) Now, these two equations represent the same line when, and only when, each of them represents the diagonal referred to. Hence, comparing the coefficients of (4) and (5), we have

$$\frac{2+5\lambda}{\mu-1}=\frac{1+\lambda}{1}=\frac{3}{\mu-2}.$$

(*iv*) Solving these equations for λ and μ, we have $\lambda=\frac{1}{2}$, $\mu=4$; and substituting the value of λ in (4) [or of μ in (5)], we have

$$y-3x-2=0$$

as the equation of the diagonal referred to. Similarly, the equations of the other diagonals can be obtained. [42]

14·3. The method of making homogeneous.

Problems in which it is required to find the equation of the straight lines joining the origin to the points of intersection of two given curves can be easily solved by combining them in such a manner (without transgressing the laws of Algebra) as might give rise to a homogeneous equation of the desired degree. [43] Thus :—

(I). *To find the equation of the straight lines joining the origin to the points of intersection of* $ax^2+2hxy+by^2+2fy+2gx+c=0$ and $y=mx+k$.

---

[42] *A complete quadrilateral has three diagonals.* If $ABCD$ be the quadrilateral in the ordinary sense of the term, its diagonals are AC BD and the line joining the point of intersection of AB, DC with that of AD, BC.

[43] If, without transgressing the laws of Algebra, we combine two equations in any manner, then the resulting equation represents a locus through the points of intersection of the loci represented by the given equations. [We should, in this case, proceed in such a manner that the degree of the resulting equation may not be greater than (it should be equal to) the number of solutions (real or imaginary) common to the two given equations]. And, by Ex. 13 of Art. 9, we know that a homogeneous equation of the nth degree in x, y represents n straight lines (real or imaginary). Hence the result.

(*i*) From $y = mx + k$, we have $\frac{y - mx}{k} = 1$.

Now, substituting a suitable power of $\frac{y - mx}{k}$ (which $= 1$) as a factor of as many terms of the other equation as will make the resulting equation homogeneous in x, y and of the desired degree, we have

$$k^2(ax^2 + 2hxy + by^2) + 2k(fy + gx)(y - mx) + c(y - mx)^2 = 0. \quad \ldots \quad (6)$$

(*ii*) Now, (6) is a homogeneous equation of the second degree; and it is clear that there are two such lines (one corresponding to each point of intersection of the two given curves). Hence, (6) is the equation of the pair of straight lines referred to.

** (II) *To find the equation of the straight lines joining the origin to the points of intersection of the curves* $ax^2 + by^2 = d$, *and* $px^2 + qy^2 + rx + sy = 1$.

From the first equation, we have $\frac{ax^2 + by^2}{d} = 1$.

$$\therefore \quad px^2 + qy^2 + rx + sy = \frac{ax^2 + by^2}{d}.$$

$$\text{or} \quad \left(p - \frac{a}{d}\right)x^2 + \left(q - \frac{b}{d}\right)y^2 = -\left(rx + sy\right).$$

From this and $ax^2 + by^2 = d$, we have

$$ax^2 + by^2 = d\left\{\frac{\left(p - \frac{a}{d}\right)x^2 + \left(q - \frac{b}{d}\right)y^2}{rx + sy}\right\}^2.$$

$$\therefore \quad (ax^2 + by^2)(rx + sy)^2$$

$$= d\left\{\left(p - \frac{a}{d}\right)x^2 + \left(q - \frac{b}{d}\right)y^2\right\}^2, \ldots (7)$$

a homogeneous equation of the fourth degree And as there are evidently 2×2 such lines (one corresponding to each point of intersection of the two curves), it is clear that (7) is the equation of the straight lines referred to.

14·4. The method of reducing to a locus problem.[44]

(I) *To find the equation of the straight line which bisects the join of* $P \equiv (x_1, y_1)$ *and* $Q \equiv (x_2, y_2)$ *at right angles.*

(*i*) The required straight line may be regarded as the locus of a point $A \equiv (x, y)$ which moves so that its distances from (x_1, y_1) and (x_2, y_2) are the same (see Ex. 7 of Art. 8).

(*ii*) Now, expressing the condition that AP may be equal to AQ, we have

$$\sqrt{(x-x_1)^2+(y-y_1)^2} = \sqrt{(x-x_2)^2+(y-y_2)^2}.$$

$$\therefore\ x_1^2+y_1^2-2xx_1-2yy_1 = x_2^2+y_2^2-2xx_2-2yy_2. \quad \ldots (8)$$

Hence, (8) is the equation of the required st. line.

(II) *To find the equations of the bisectors of the angles between the straight lines* $x \cos \alpha + y \sin \alpha - p = 0$ *and* $x \cos \alpha' + y \sin \alpha' - p' = 0$.

(*i*) The bisector of an angle between two straight lines may be regarded as the locus of a point $P \equiv (x, y)$ which moves so that its perpendicular distances from the given straight lines are numerically equal.

[44] See Exs. 7 and 8 of Art. 8—the two locus problems which have been solved in Ch. I.

(*ii*) Hence, by Art. 12·1, we have

$$x \cos \alpha + y \sin \alpha - p = \pm (x \cos \alpha' + y \sin \alpha' - p') \quad \dots \quad (9)$$

as the equations of the bisectors.

[If the + sign is taken in (9), then $x \cos \alpha + y \sin \alpha - p$ and $x \cos \alpha + y \sin \alpha - p$ must be of the same sign. Hence, P must be on the same side of each of the two lines as the origin, or on the opposite side of both of them. In this case, it will be the bisector of that angle between the given straight lines in which the origin lies. This can be easily verified with the help of a diagram representing the state of affairs. If the − sign is taken in (9), then we shall have the equation of the bisector of the other angle.]

14·5. The method of discussing the roots of a quadratic.

In some cases, the equation of a straight line *connected* with a curve can be obtained by a process whose characteristic feature is the discussion of the roots of a quadratic equation.

For an illustrative example, see Art. 18·2 (i), Ch. III.

**14·6. The method of verification.

By assuming a certain equation in an abrupt manner, we may be able to verify that it satisfies the conditions of the problem under consideration. The difficulty of applying this method is clear from the fact that it depends on our ability to anticipate the result (or on a previous knowledge of it). Thus :—

To find the equation of the straight line passing through (x_1, y_1) and (x_2, y_2).

Consider the equation $\dfrac{x-x_1}{x_2-x_1}=\dfrac{y-y_1}{y_2-y_1}$.

Being an equation of the first degree, it represents a straight line. It passes through (x_1, y_1) as the equation is evidently satisfied when x_1, y_1 are written for x and y respectively. Similarly, it can be seen that it passes through (x_2, y_2). Hence, the above equation represents the required straight line.

For another example, see Art. 191, Ch. III.

Examples.

1. Find the equation of the straight line which passes through (2, 4) and is perpendicular to $5x+7y=1$.

2. Find the equation of the straight line which cuts off an intercept 5 from the axis of x and is parallel to $5x+7y=1$.

3. Find the equation of the straight line which passes through the origin and is perpendicular to $5x+7y=1$.

4. Find the equation of the straight line which passes through (2, 1) and makes equal angles with the axes.

5. Show that the equation of any straight line passing through $P\equiv(x_1, y_1)$ can be written as

$$\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta},$$

where θ is the angle which the upward direction of the straight line makes with the axis of x.[45-46]

[45] Each of the above ratios $=\dfrac{\sqrt{(x-x_1)^2+(y-y_1)^2}}{\sqrt{\cos^2\theta+\sin^2\theta}}=r$, where r is the algebraic distance of any point $Q\equiv(x, y)$ on the straight line from P.

6. **Find the equation of the straight line which passes through the intersection of $3x+4y-1=0$ and $2x+y+5=0$, and makes an angle of $45°$ with the axis of y.**

7. Find the equation of the straight line joining $(3, 4)$ to the intersection of the lines $3x+y=5$ and $5x-7y+1=0$.

8. Find the equations of the diagonals of the parallelogram the equations of whose sides are $x=2$, $y=x+1$, $x=6$, $y=x+3$.

9. Find the equations of the diagonals of the parallelogram the equation of whose sides are $ax+by\pm c=0$, $a_1x+b_1y\pm c=0$.

10. Find the equation of the straight line joining the origin to the intersection of $ax+by=c$ and $a'x+b'y=c'$.

11. Find the equation of the straight lines joining the origin to the points common to the curves

$$x^2+3y^2+4xy-x+5y=0 \text{ and } x+y+1=0.$$

12. Find the equation of the straight lines joining the origin to the points of intersection of the line $ax+by+c=0$ with the lines $lx+my+n=0$ and $px+qy+r=0$.

13. Find the equations of the straight lines joining the origin to the points of intersection of the curves $x^2+y^2=r^2$ and $ax^2+by^2+2hxy+2fy+2gx+c=0$.

14. Find the equation of the straight line which divides the line joining $(2, 3)$ and $(-5, 4)$ in the ratio of $4:7$, and is perpendicular to it.

Hence, the co-ordinates of any such point Q can be written a $(x_1+r\cos\theta, y_1+r\sin\theta)$.

[40] We say '*upward direction*' because of the geometrical significance of the 'm' of a straight line.

15. Find the polar equation of the straight line passing through $P \equiv (r_1, \theta_1)$ and $Q \equiv (r_2, \theta_2)$.

16. Find the equations of the bisectors of the angles between the st. lines $ax + bx + c = 0$ and $a'x + b'y + c' = 0$.

[*Hints.* Reduce them to their normal forms, and then proceed as in Art. 14·4 (II).]

17. Find the equation of the bisector of that angle between the straight lines $3x + 2y = 1$ and $y = 5x + 7$ in which the origin lies.

18. Find the equation of the lines bisecting the angles between the lines $ax^2 + 2hxy + by^2 = 0$.

[*Solution.* Let $ax^2 + 2hxy + by^2 \equiv b\,(y - m_1 x)\,(y - m_2 x)$.

Then, $m_1 + m_1 = -\dfrac{2h}{b}$ and $m_1 m_2 = \dfrac{a}{b}$.

Now, the bisectors are evidently given by

$$\frac{y - m_1 x}{\sqrt{1 + m_1{}^2}} \pm \frac{y - m_2 x}{\sqrt{1 + m_2{}^2}} = 0.$$

Multiplying the two equations, we have

$$\frac{(y - m_1 x)^2}{1 + m_1{}^2} - \frac{(y - m_2 x)^2}{1 + m_2{}^2} = 0.$$

$$\therefore \quad (y^2 - x^2)\,(m_2 + m_1) + 2(1 - m_1 m_2)xy = 0,$$

$$\text{or } \frac{x^2 - y^2}{a - b} = \frac{xy}{h} \qquad \ldots \qquad \ldots \qquad (10)$$

as the equation of the bisectors.] [47]

[47] Writing (10) as $\left(\dfrac{y}{x}\right)^2 + \dfrac{a-b}{h} \cdot \dfrac{y}{x} - 1 = 0$, and noting that the roots of this equation in $\dfrac{y}{x}$ are *real*, we conclude that the bisectors are *real* straight lines *even when the given lines are imaginary*.

**19. If the general equation of the second degree represents two straight lines, find the equation of the bisectors of the angles between these lines.

*20. Find the equation of the straight line which passes through (2, 3) and is perpendicular to $3x+4y-7=0$, the axes being inclined at an angle of 60°.

*21. Find the equation of the straight line which cuts off an intercept 4 on the axis of x and is inclined to it at an angle of 45°, the angle between the axes being 60°.

15.

ON SOME LOCUS PROBLEMS CONNECTED WITH STRAIGHT LINES. [48]

15·1. The direct method. [49]

By assuming the co-ordinates of the moving point to start with, the equation of the locus can sometimes be written down at once (with the help of the standard results and formulæ). Thus:—

[48] In this article, we shall discuss some problems on the locus of a point in which a knowledge of the Analytical Geometry of the straight line is *sufficient*, although the locus found will *not necessarily* be a straight line. In fact, it is not generally possible for us to state whether a certain locus will be a straight line or not until we have actually found its equation.

It need hardly be stated that the equation of the locus of a point which moves in an assigned manner is a relation between the co-ordinates of the moving point, and is free from any other variable quantity (the relation must, of course, be consistent with the conditions of the problem).

[49] Note that the classification (which is based on experience) has the same purpose in view as that stated in foot-note 39 in connection with the construction of equations of straight lines satisfying assigned conditions. It *will give a rough idea of things so far as locus problems discussed in this book are concerned.*

To find the locus of a point which moves so that the sum of the squares of its distances from (2, 3) *and* (1, −2) *may be equal to the square of its distance from* (7, 5).

(*i*) Let $P \equiv (x, y)$ be the moving point.

(*ii*) Then, expressing the geometrical condition stated above analytically, we have

$$(x-2)^2+(y-3)^2+(x-1)^2+(y+2)^2 = (x-7)^2+(y-5)^2.$$
$$\therefore \quad x^2+y^2+8y+8x=56.$$

Now, as this is a relation between the co-ordinates of any point on the locus, and as it does not contain any other variable quantity, it is the equation of the required locus.

15·2. The elimination method.

There are some types of problems in which the equation of the locus cannot be written down at once (*i.e.*, as easily as in Art. 15·1), but its equation can be found by eliminating certain variable quantities from a sufficient number of equations. This method can be very usefully applied in case of problems of the following types :—

(I) *A line of constant length slides between two fixed straight lines which include a right angle. Find the locus of its middle point.*

Take the fixed lines as the axes, and let $\frac{x}{a}+\frac{y}{b}=1$ be the equation of the moving line. Now, representing the state of affairs by means of a diagram (fig. 23), we have $A \equiv (a, o)$, $B \equiv (o, b)$.

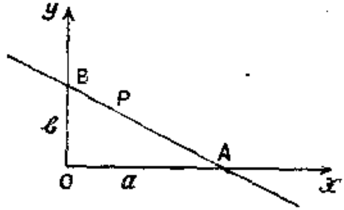

FIG. 23

Hence, if $P \equiv (x, y)$ be the middle point of AB, we must have

$$x = \frac{a}{2}, \qquad \ldots \qquad (2)$$

$$\text{and } y = \frac{b}{2}. \qquad \ldots \qquad (3)$$

And since AB is of constant length, we have

$$a^2 + b^2 = k^2 \text{ (say)}. \qquad \ldots \qquad (4)$$

Eliminating a and b (which vary with the change of the position of AB) from these three equations, we have

$$x^2 + y^2 = \frac{k^2}{4}$$ as the equation of the locus.[50]

(II) *P, P' are two points on the x-axis, and Q, Q' are two points on the y-axis such that $OP + OQ = OP' + OQ'$. If P and Q are fixed, find the locus of the point of intersection of PQ' and $P'Q$.*

Let $P \equiv (a, o)$, $Q \equiv (o, b)$, $P' \equiv (\alpha, o)$ and $Q' \equiv (o, \beta)$.

Then, by the condition of the problem, we have

$$a + b = \alpha + \beta. \qquad \ldots \qquad (5)$$

The equation of PQ' is

$$\frac{x - a}{-a} = \frac{y}{\beta}; \qquad \ldots \qquad (6)$$

and that of $P'Q$ is

$$\frac{x - \alpha}{-\alpha} = \frac{y}{b}. \qquad \ldots \qquad (7)$$

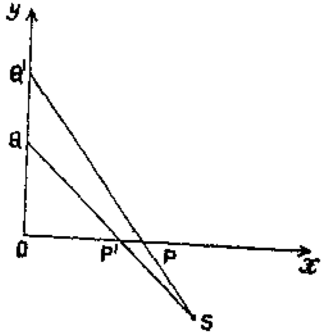

FIG. 24

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[50] If the fixed lines were not at right angles to each other, the same procedure would help us in solving the problem. In that case (4) would be replaced by the relation of $a^2 + b^2 - 2ab \cos \omega = k^2$, ω being the angle between the axes.

Now, (6) and (7) hold together at the point of intersection of these lines; and the other variable parameters α and β occurring in these equations are connected by the relation (5). Hence, the equation of the locus is obtained by *eliminating* α and β from (5), (6) and (7).

To eliminate α and β, we proceed as follows :—

From (6) and (7), we have

$$\beta = \frac{y}{1-\frac{x}{a}} \text{ and } \alpha = \frac{x}{1-\frac{y}{b}} \text{ respectively.}$$

Substituting these values of α and β in (5), we have

$$a+b=\frac{x}{1-\frac{y}{b}}+\frac{y}{1-\frac{x}{a}}.$$

This is, therefore, the equation of the locus.

15·3. The method of polar co-ordinates.

Certain locus problems (specially those in which the locus of the end point of a variable line passing through a fixed point is required) can be very conveniently solved by making use of polar co-ordinates (instead of Cartesian co-ordinates), or by using Cartesian co-ordinates and applying our knowledge of polar co-ordinates at a suitable stage of the work. Thus :—

(I) *Given a fixed point O and two fixed straight lines P and Q. If through O any straight line be drawn meeting the fixed lines at P and Q, and a point R be taken on it so that*

$$\frac{2}{OR}=\frac{1}{OP}+\frac{1}{OQ},$$

then find the locus of R.

Let the equations of the fixed lines be $r\cos(\theta-\alpha_1)=p_1$ and $r\cos(\theta-\alpha_2)=p_2$ with reference to O as the *pole* and any line through it as the *initial line.*

Now, the equation of any line through O is $\theta=\theta_1$. If it cuts the given lines at P and Q, then we have

$$OP=\frac{p_1}{\cos(\theta_1-\alpha_1)} \text{ and } OQ=\frac{p_2}{\cos(\theta_1-\alpha_2)}.$$

Hence, by the condition of the problem, we have

$$\frac{2}{r_1}=\frac{\cos(\theta_1-\alpha_1)}{p_1}+\frac{\cos(\theta_1-\alpha_2)}{p_2},$$

writing r_1 for OR.

This is a relation between the polar co-ordinates of any such point $R\equiv(r_1, \theta_1)$, and is free from any other variable parameter. Hence, the polar equation of the locus is

$$\frac{2}{r}=\frac{\cos(\theta-\alpha_1)}{p_1}+\frac{\cos(\theta-\alpha_2)}{p_2}.$$ [51-52]

(II) *From a given point A perpendiculars are drawn to straight lines passing through another point B; find the locus of the feet of these perpendiculars.*

Let $x\cos\alpha+y\sin\alpha-p=0$ be the equation of any straight line with reference to any rectangular axes through A. If it passes through $B\equiv(h, k)$, we have

$$h\cos\alpha+k\sin\alpha-p=0.$$

Now, p and α may be regarded as the polar co-ordinates of the foot of the perpendicular from A

[51] Replacing r_1, θ_1 by the current co-ordinate r, θ of any point on the locus, we have this equation from the previous one.

[52] By changing to Cartesian co-ordinates, it can be easily seen that the locus is a *straight line.*

to the above straight line. And as they are connected by the relation $h \cos \alpha + k \sin \alpha - p = 0$ which contains no other variable parameter, the polar equation of the locus is

$$h \cos \theta + k \sin \theta - r = 0.$$

15·4. The parametric method.

The characteristic feature of this method is that the number of variable parameters to be *eliminated* is reduced to a minimum (or very nearly to it) by expressing the Cartesian co-ordinates of each point on a curve in terms of a single parameter. [This makes the task of elimination very easy more often than not.]

For an illustrative example, see Art. 23·2, Ch. III.

Examples.

1. A point moves so that the sum of its perpendicular distances from two given straight lines is equal to a constant. Show that its locus is a straight line.

2. A point moves so that the sum of its perpendicular distances from five given straight lines is equal to a constant. Show that its locus is a straight line.

3. Find the locus of the vertex of a triangle when the base and the sum of the squares of its sides are given.

4. Any straight line is drawn through a fixed point A cutting the axes (through O) in the points P and Q. Find the locus of R which is such that $OPRQ$ is a rectangle.

5. A variable straight line cuts the axes (through O) in P and Q such that the area of the triangle OPQ is equal to a constant. Find the locus of the middle point of the portion of the straight line intercepted between the axes.

6. OPA, OQB are any two fixed straight lines perpendicular to each other. If P and Q are fixed points, and A and B are such that PA/QB = constant, find the locus of the middle point of AB.

**7. In Ex. (II) of Art. 15·2, find the locus of the intersection of PQ' and $P'Q$ when

$$\frac{1}{OP'} - \frac{1}{OQ'} = \frac{1}{OP} - \frac{1}{OQ}.$$

8. In Ex. 6, find the locus of the point of intersection of AQ and BP.

9. Through a given point O a straight line is drawn to cut two given straight lines in P and Q; find the locus of a point R on this variable straight line in the following cases :—

(i) when OP, OR and OQ are in $A.P.$

(ii) when OP, OR and OQ are in $G.P.$

10. A line of constant length slides between two fixed straight lines which cut at right angles at O; find the locus of the foot of the perpendicular from O on the given line.

11. From a given point O a perpendicular is drawn to any straight line through another fixed point. If P be the foot of this perpendicular, and OP is produced to Q so that $\frac{OP}{OQ}$ = constant, find the locus of Q.

12. One vertex of a triangle is fixed and another moves along a fixed straight line; find the locus of the third when the angles are given.

[*Solution.* Let A be the fixed vertex and BD be the fixed line. Draw AD perpendicular to BD (fig. 25); and let $AD = a$, a known quantity. Also let $C \equiv (r, \theta)$ with reference to A as the *pole* and AD as the *initial line*. Then since the angles are given, we have

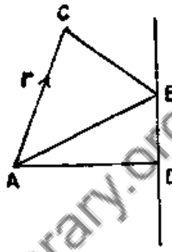

FIG. 25

$$\frac{AB}{AC} = \frac{\sin C}{\sin B} = \text{constant} = K \text{ (say)}.$$

Hence, $a = AD = AB \cos DAB$
$= K.AC \cos (\theta - A)$
$= Kr \cos (\theta - A)$.

This is a relation between the co-ordinates of $C \equiv (r, \theta)$, and is free from any other variable quantity. So, the polar equation of the locus is

$$Kr \cos (\theta - A) = a,$$

which is a straight line.]

**13. In an equilateral triangle ABC, find the locus of A, if B is fixed and C lies on a fixed straight line.

16.

MISCELLANEOUS TOPICS CONNECTED WITH STRAIGHT LINES.[53]

16·1. On problems in which it is required to show that a variable straight line passes through a fixed point.

In such cases, one of the methods is (i) to find the equation of the variable straight line, and (ii) write it as

$$ax + by + c + \lambda(a'x + b'y + c') = 0, \quad \dots \quad \dots \quad (1)$$

where λ is an undetermined multiplier and $a, b, c, \; a', b', c'$

[53] In this section, we shall discuss some of the well-known topics which have not been discussed before.

are known constants. If this is done, we can at once conclude that (1) passes through the fixed point common to the straight lines $ax+by+c=0$ and $a'x+b'y+c'=0$.[54]

The following is an illustrative example :—

A straight line moves in such a manner that the sum of the reciprocals of its intercepts on the axes is a constant ; show that it always passes through a fixed point.

Let the equation of the moving straight line be

$$\frac{x}{a}+\frac{y}{b}=1. \quad \ldots \quad (2)$$

Then, by the condition of the problem, we have

$$\frac{1}{a}+\frac{1}{b}=\frac{1}{k} \text{ (say)}. \quad \ldots \quad (3)$$

From (3), we have $\frac{1}{b}=\frac{1}{k}-\frac{1}{a}$.

Now, substituting this value of $\frac{1}{b}$ in (2),[55] we have

$$\frac{x}{a}+y\left\{\frac{1}{k}-\frac{1}{a}\right\}=1.$$

$$\therefore \qquad \frac{y}{k}-1+\frac{1}{a}(x-y)=0,$$

where $\frac{1}{a}$ is an undetermined multiplier.

Hence, the straight line passes through the fixed point common to the lines $x-y=0$ and $y=k$.

---

[54] This is clear from the fact that (i) is satisfied whenever $ax+by+c=0$ and $a'x+b'y+c'=0$ are simultaneously satisfied.

[55] This is equivalent to eliminating one of the parameters a, b from (2) and (3).

16·2. On the area of a triangle the equations of whose sides are given.

In such cases, the best (though not necessarily the simplest) thing is to reduce the problem to one in which the co-ordinates of the angular points are given (*cf.* Art. 6).[56] Thus :—

To find the area of the triangle formed by $x+y=2$, $3x-2y+5=0$ *and* $x=0$.

Solving any two equations at a time, we see that the angular points of the triangle are $(0, 2)$, $(0, \frac{5}{2})$ and $(-\frac{1}{5}, \frac{11}{5})$.

Hence, by Art. 6·3, the area of the triangle

$$=\frac{1}{2}\begin{vmatrix} 0 & 2 & 1 \\ 0 & \frac{5}{2} & 1 \\ -\frac{1}{5} & \frac{11}{5} & 1 \end{vmatrix}=\frac{1}{20}.$$

****16·3. On theorems which can be proved by verifying the truths of simpler but equivalent theorems.**

(I) *Show that the straight lines* $10x^2+7xy+3y^2=0$ *are equally inclined to the straight lines* $7x_2+10xy-3y^2=0$.

This will be so, if the two pairs of lines have the same bisectors.

Now, the equation of the bisectors are given by $x^2-y^2=2xy$ in each case. Hence the theorem.

(II) *Show that the straight lines given by* $2x+3y=4$ and $(2x+3y)^2-3(2y-3x)^2=0$ *form an equilateral triangle.*

---

[56] A method of finding the area of a triangle without actually finding the co-ordinates of the vertices is given in Askwith's *Conic Sections*, Ch. III.

This will be so, if the acute angle between the first line and each of the other lines is 60°.

Now, the 'm' of the first line $= -\frac{2}{3}$; and that of one of the other lines is

$$\frac{2+3\sqrt{3}}{2\sqrt{3}-3}.$$

Hence, the tangent of the angle between them $= \pm\sqrt{3}$. Hence, the acute angle between them is 60°. Similarly, it can be shown that the other acute angle is also equal to 60°. Hence, the triangle is equilateral.

Examples.

1. In an isosceles triangle ABC, the equal sides AB, AC are produced to D and E such that $\dfrac{BD}{AB} = \dfrac{AB}{CE}$; show that DE passes through a fixed point.

2. If a straight line be such that the sum of the perpendiculars on it from three given points is zero, it passes through a fixed point.

3. Find the area of the triangle formed by the straight lines whose equations are $y=2x$, $y=4x$ and $y=7x+5$.

4. Find the area of the triangle formed by the straight lines whose equations are

$$y_r = m_r x + c_r \ (r=1, 2, 3).$$

5. Find the area of the parallelogram formed by the straight lines whose equations are $2x-3y=\pm 4$ and $3x-2y=\pm 1$.

**6. Show that the straight lines $ax^2+2hxy+by^2=0$ are equally inclined to the straight lines $(a-\lambda)x^2+2hxy+(b-\lambda)y^2=0$ for all values of λ

**7. Find the condition that the pair of straight lines $x^2+2hxy-y^2=0$ may bisect the angle between the pair of lines $x^2+2kxy-y^2=0$, and vice versa.

**8. Show that the straight lines given by $(y-2x)^2=45$ and $(y-3x)^2=90$ form a *rhombus*.

9. Show that the equation $x^2+2xy+y^2-4x-4y-5=0$ represents two parallel straight lines. Also find the perpendicular distance between them.

**10. Show that the straight lines $(a^2-3b^2)x^2+8abxy+(b^2-3a^2)y^2=0$ form an equilateral triangle with the straight line $ax+by+1=0$.

11. Find the orthocentre of the triangle formed by the straight lines of Ex. 3.

12. Find the centroid of the triangle formed by the straight lines of Ex. 3.

CHAPTER III.

THE CIRCLE.

17.

ON THE CIRCLE AND ITS EQUATION.

17·1. Definitions.

A circle is the locus of a point P which moves so that its distance from a fixed point C is always the same. The fixed point is called the *centre* of the circle, and the constant value of the distance is called its *radius*.

17·2. On the general form of the equation of the circle (the axes being rectangular).

Let the centre of the circle be at $C \equiv (x_1, y_1)$, and let its radius be a; also let $P \equiv (x, y)$ be any point on the circle. Then, by definition, its equation is

$$\sqrt{(x-x_1)^2+(y-y_1)^2}=a,$$

$$\text{or} \qquad (x-x_1)^2+(y-y_1)^2=a^2. \qquad \ldots \qquad (1)$$

From the above method of procedure, it is clear that the equation of a circle can *always* be written in this form.

17·21. Its standard form.

Let the centre of the circle be taken as the origin, and let two mutually perpendicular lines through it be taken as the axes of co-ordinates ; also let $P \equiv (x, y)$ be any point on the circle.

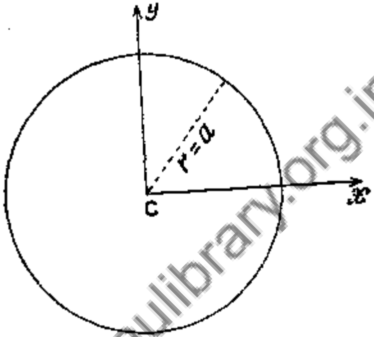

FIG. 26

Then, by definition, its equation is

$$x^2 + y^2 = a^2, \quad \ldots \quad (2)$$

a being its radius. This is known as the *standard form* of the equation of the circle.[1-2]

17·22. Every equation of the form

$$x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0 \quad \ldots \quad \ldots \quad (3)$$

represents a circle (the axes being rectangular).

Re-arranging the terms in (3), we have

$$(x + g_1)^2 + (y + f_1)^2 = g_1^2 + f_1^2 - c_1. \quad \ldots \quad \ldots \quad (4)$$

Comparing (4) with (1), we see that (3) represents a circle whose centre is at $(-g_1, -f_1)$ and radius $= \sqrt{g_1^2 + f_1^2 - c_1}$.

17·3. To find the conditions that the general equation of the second degree in x and y may represent a circle.

The most general equation of the second degree in x, y is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, \quad \ldots \quad \ldots \quad (5)$$

where a, h, b, g, f and c are arbitrary constants (none being *zero*).[3]

[1] Note that (2) could have been obtained from (1) by changing the origin to (x_1, y_1) without changing the directions of the axes.

[2] As the equation of the circle can always be reduced to this form, it is clear that any geometrical property of the circle deduced from (or proved with the help of) this equation may be regarded as having been deduced (or proved) for circles in general.

[3] It includes all equations of the second degree in x, y as special cases.

Comparing (5) with (3), we have

$$\frac{a}{1}=\frac{b}{1}=\frac{h}{0}=\frac{f}{f_1}=\frac{g}{g_1}=\frac{c}{c_1}.$$

From the first two equations of this set, we have $a=b$ and $h=0$. These conditions must, therefore, be satisfied in order that (5) may represent a circle. (They are *necessary* for the purpose).

17·31. The conditions $a=b$ and $h=0$ are sufficient to make (5) represent a circle.

Putting $a=b$ and $h=0$ in (5), and dividing throughout by a, we have

$$x^2+y^2+2\frac{f}{a}y+2\frac{g}{a}x+\frac{c}{a}=0.$$

$$\therefore \quad \left(x+\frac{g}{a}\right)^2+\left(y+\frac{f}{a}\right)^2=\frac{g^2+f^2-ac}{a^2}.$$

Hence, (5) represents a circle having its centre at $\left(-\frac{g}{a}, -\frac{f}{a}\right)$ and radius $=\frac{\sqrt{g^2+f^2-ac}}{a}$.

Hence, the conditions $a=b$ and $h=0$ are *sufficient* to make (5) represent a circle.

17·32. Note on the conditions necessary to determine a circle.

The equation of a circle contains three independent constants. Hence, *three conditions are necessary and sufficient to determine a circle,* these conditions being such that each may give rise to one independent equation between the constants. Thus, a circle may be made to pass through three given points (not in one straight line), or pass through two given points and touch a given straight line. [4]

17·4. Note on some points connected with the circle.

We have seen that when (5) represents a circle, its radius $=\dfrac{\sqrt{g^2+f^2-ac}}{a}$. Now,

(i) If $g^2+f^2=ac$, the radius of the circle vanishes.

In this case, the circle degenerates into a point coinciding with its centre. Hence, *the point may be regarded as a special case of the circle.*

(ii) If $g^2+f^2>ac$, the radius will be *finite or infinitely large* according as *a is not equal to or equal to zero.*

In the first case, it is the circle in the ordinary sense of the term. In the second case, (5) reduces to $2gx+2fy+c=0$ which is a straight line. Hence, *the straight line may also be regarded as a special case of the circle.*

(iii) If $g^2+f^2<ac$, the circle is *imaginary.*

17·5. Polar equation of the circle.

Let the centre C of the circle be taken as the pole, and let any line Cx through it be taken as the *initial line* (fig. 26). Then, by definition, the equation of the circle is

$$r=a,$$

a being its radius.[5]

[4] Any condition which gives rise to two (or more) equations between the constants must, for this purpose, be regarded as equivalent to two (or more) conditions. Moreover, more than one circle may be found to satisfy a given set of sufficient conditions.

[5] This equation could have been obtained from (2) with the help of the usual formulæ of transformation of Art. 3·1.

* 17·6. The position in oblique co-ordinates.

From the definition of a circle, it is clear that the equation of a circle may be written as

$(x-x_1)^2+(y-y_1)^2+2(x-x_1)(y-y_1)\cos\omega=a^2$, ω being the angle between the axes, and other quantities having the same significance as in Art. 17·2.

If we take the origin at the centre of the circle, the equation becomes $x^2+y^2+2xy\cos\omega=a^2$.

17·7. Parametric equation of a circle.

Putting $x=a\cos\theta$ in $x^2+y^2=a^2$, we have

$y=a\sin\theta$ as a solution.

This shows that $(a\cos\theta, a\sin\theta)$ is a point on the circle $x^2+y=a^2$ for all values of θ (one point for each value of θ). For this reason, the equations

$$\left.\begin{aligned} x&=a\cos\theta,\\ \text{and } y&=a\sin\theta.\end{aligned}\right\}\quad\ldots\qquad(6)$$

are called the *parametric* equations of the circle $x^2+y^2=a^2$. [6-7]

Examples.

1. Find the equation of the circle whose centre is at (3, 4) and radius = 5.

---

[6] The equations (6) express, in a suitable manner, the co-ordinates of any point on a circle in terms of a single variable parameter; but they do not constitute the only way of doing so. This is clear from the procedure followed above. In fact, there is no limit to the number of ways of doing so. The above mode of representing the co-ordinates of any point on a circle in terms of a single variable parameter has, however, been found to be very useful in many cases.

[7] A point whose co-ordinates are given by (6), is usually called the point "θ" on the circle whose equation is $x^2+y^2=a^2$ (or simply the point θ).

2. The points (2, 3) and (−4, 7) are the end points of a diameter of a circle; find its equation.

3. Find the equation of the circle which has $A \equiv (x', y')$ and $B \equiv (x'', y'')$ as the end points of a diameter.

[*Hints.* The equation of the circle can be obtained by finding the centre and the radius, or by regarding it as the locus of a point P which moves so that the $\angle APB$ is a right angle.]

** 4. Find the equation of the circle which passes through (2, 3), (3, 4) and (−2, 3).

** 5. Find the equation of the circle which passes through the origin, and cuts off intercepts equal to 4 and 5 from the axes.

[*Solution.* The equation of any circle passing through the origin is $x^2 + y^2 + 2gx + 2fy = 0$.

It cuts $y = 0$ in points given by $x^2 + 2gx = 0$. The roots of this equation are 0 and $-2g$. Hence, the intercept on the x-axis $\equiv -2g$ as can be easily seen from a diagram. Hence, by the condition of the problem, we have $-2g = 4$, $\therefore g = -2$. Similarly, since the intercept on the y-axis $\equiv -2f = 4$, we have $f = -2$. Hence, the equation of the circle is $x^2 + y^2 - 4y - 4y = 0$.]

**6. Find the equation of the circle which touches (cuts in two coincident points) the axis of x at a distance 4 from the origin, and cuts off an intercept 6 from the other axis.

**7. Find the equation of the circle which touches both the axes and passes through the point (−3, −4).

8. Trace the following curves :—

(1) $x^2+y^2-2ax=0.$

(2) $r=a \sin\theta.$

(3) $x^2+y^2-2x-4y=14.$

9. Find the polar equation of a circle of radius a with reference to a point in the circumference as the pole, and the diameter through it as the initial line.

[*Hints.* This can be easily done by using the fact that the angle in a semi-circle is a right angle; or by finding the Cartesian equation, and then applying the formulæ of transformation of Art. 31.]

10. Find the centre and radius of the circle

$$r=a\cos\theta+b\sin\theta.$$

[*Hints.* Change to the Cartesian system, and proceed as usual.]

11. Show that $(a \tanh\theta,\ a \operatorname{sech}\theta)$ is a point on the circle $x^2+y^2=a^2$ for all values of θ.

12. Show that the point $P\equiv(x', y')$ will lie *within*, *on*, or *without* the circle $x^2+y^2=a^2$ according as $x'^2+y'^2$ $<, =,$ or $>a^2$.

[*Solution.* If $x'^2+y'^2<a^2$, the distance of the point P from the centre of the circle is less than its radius. Hence, P must lie within the circle. Similarly, the truth of the other results can be seen.]

*13. If the axes are inclined at an angle ω, then the necessary and sufficient conditions that

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

may represent a circle are $a=b$ and $h=a\cos\omega$.

[*Hints*. Comparing it with $(x-x_1)^2+(y-y_1)^2$ $+2(x-x_1)(y-y_1)\cos\omega=a^2$, we find the conditions to be *necessary*. To prove that they are also *sufficient*, we proceed backwards as in Art. 17·31.]

*14. Find angle between the axes so that x^2+y^2-xy $+2x+2y+1=0$ may represent a circle.

*15. Does the equation $x^2+y^2=a^2$ represent a circle even when the axes are oblique ?

18.

ON TANGENTS.

18·1. Definitions.

Every straight line lying in the plane of a circle cuts it in two points, real or imaginary.[8] Further, these points of intersection may be distinct or coincident. Any line which meets a circle in two coincident points is called a *tangent* to it, and the point of coincidence is called the *point of contact* of the tangent and the circle.[9]

18·2. To find the equation of the tangent at any point (x_1, y_1) on the circle $x^2+y^2=a^2$. ... (1)

(i) *By the method of discussing the roots of a quadratic.*

---

[8] For, a straight line is given by an equation of the first degree in x and y, and a circle by one of the second degree; and these equations have two common solutions, real or imaginary.

[9] These terms may be defined similarly in case of any curve of the second degree, *i.e.*, any curve represented by an equation of the second degree in x and y.

If P and Q be the points where any line meets a circle (or a curve) of the second degree, then PQ is called a *chord* of the circle (or of the curve). Hence, a *tangent* may be regarded as the limiting case of a *chord*.

The equation of a straight line through $P\equiv(x_1, y_1)$ is

$$\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r, \quad \ldots \quad \ldots \quad (2)$$

where r is the algebraic distance of any point $Q\equiv(x, y)$ on it from P, and θ = the angle which the upward direction of PQ makes with the axis of x. The point Q is given by

$$x=x_1+r\cos\theta \text{ and } y=y_1+r\sin\theta.$$

If it lies on the given circle (1), we must have

$$(x_1+r\cos\theta)^2+(y_1+r\sin\theta)=a^2.$$

$$\therefore r^2+2r(x_1\cos\theta+y_1\sin\theta)+x_1{}^2+y_1{}^2-a^2=0. \quad \ldots \quad (3)$$

The roots of this quadratic equation in r give the distances from P of the points of intersection of (1) and (2). Now, if the straight line touches the circle at $P\equiv(x_1, y_1)$, both the roots of r in (3) must be equal to zero. The conditions for this are

$$x_1{}^2+y_1{}^2-a^2=0 \quad \ldots \quad \ldots \quad (4)$$

$$\text{and} \quad x_1\cos\theta+y_1\sin\theta=0. \quad \ldots \quad \ldots \quad (5)$$

Eliminating θ from (2) and (5), we have

[1] If PQ be the given line, and Q_1, Q_2 be the points where it is cut by the given circle (fig. 27), then the roots of (3) are PQ_1 and PQ_2. A consideration of the state of affairs as represented in fig. 27 will tell us that

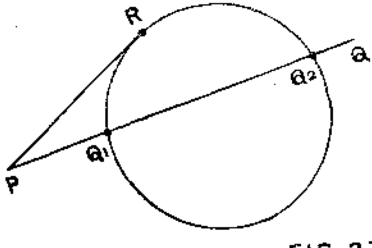

FIG. 27

(i) The straight line will touch the circle at P, if both the roots of (3) are *zero*.

(ii) The chord Q_1Q_2 will be bisected at the point P, if the roots of (3) are *equal and opposite*.

$$\frac{x - x_1}{y - y_1} = -\frac{y_1}{x_1}, \text{ each side being equal to } \cot\theta.$$

$$\therefore \; x_1(x - x_1) + y_1(y - y_1) = 0$$

which is the equation of the tangent at (x_1, y_1).[11]

To put it in another form, we write the equation as

$$xx_1 + yy_1 - (x_1^2 + y_1^2) = 0.$$

Simplifying this with the help of (4), we have

$$xx_1 + yy_1 - a^2 = 0. \quad \ldots \quad \ldots \quad (6)$$

This is the form in which the equation of the tangent is generally written.[12]

(ii) By the method of Differential Calculus.

The equation of any line through $P \equiv (x_1, y_1)$ is

$$y - y_1 = m(x - x_1). \quad \ldots \quad \ldots \quad (7)$$

If it touches the circle $x^2 + y^2 = a^2$ at (x_1, y_1), then our knowledge of the Calculus tells us that

$m =$ the value of $\frac{dy}{dx}$ at (x_1, y_1), provided that the axes are rectangular.

$$\therefore \; m = \frac{-x_1}{-y_1}.\;^{13}$$

(iii) The straight line PQ will be a tangent to the circle from the point P outside it, if the roots of (3) are *equal*.

[11] We have now an example of the use of the method of discussing the roots of a quadratic equation in constructing the equation of a straight line (Art. 14·5).

[12] Note that the method does *not* depend on any characteristic property of the circle. We may, therefore, find it useful in case of any equation of the second degree in x, y.

[13] Differentiating $x^2 + y^2 = a^2$, we have $x + y\frac{dy}{dx} = 0$.

$\therefore \; \frac{dy}{dx} = -\frac{x}{y}$. Hence, its value at $(x_1, y_1) = -\frac{x_1}{y_1}$.

Substituting this value of m in (7), we have

$$x_1(x-x_1)+y_1(y-y_1)=0$$

as the equation of the tangent.

And since (x_1, y_1) lies on (1), we have $x_1{}^2+y_1{}^2=a^2$.

Hence, the equation of the tangent simplifies into

$$xx_1+yy_1=a^2.$$ [14]

18·21. A geometrical theorem.

Any tangent to a circle is perpendicular to the radius drawn to the point of contact.

Let the equation of the circle be $x^2+y^2=a^2$.

Now, the equation of the tangent at the point $P\equiv(x_1, y_1)$ on it is $xx_1+yy_1-a^2=0$; and the equation of the radius to the point of contact P is $xy_1=yx_1$.

---

[14] When the axes are rectangular, equation of the tangent at any point (x_1, y_1) on the curve $F(x,y)\equiv ax^2+2hxy+by^2+2gx+2fy+c=0$ may be obtained as follows :—

The equation of any line through (x_1, y_1) is $y-y_1=m(x-x_1)$.

It touches the given curve $F(x, y)=0$ at (x_1, y_1), if

$$m=\text{ the value of } \frac{dy}{dx} \text{ at } (x_1, y_1)=-\frac{\delta F}{\delta x_1}\Big/\frac{\delta F}{\delta y_1}.$$

Hence, the equation of the tangent is $(x-x_1)\dfrac{\delta F}{\delta x_1}+(y-y_1)\dfrac{\delta F}{\delta y_1}=0.$

To put it another form, we introduce a suitable power of z to make $F(x, y)=0$ homogeneous in x, y and z.

Then, by Euler's Theorem, we have $x\dfrac{\delta F}{\delta x_1}+y\dfrac{\delta F}{\delta y_1}+z\dfrac{\delta F}{\delta z_1}=0.$

Using this result, we have

$$x\frac{\delta F}{\delta x_1}+y\frac{\delta F}{\delta y_1}+z\frac{\delta F}{\delta z_1}=0 \text{ (where } z=z_1=1) \text{ as the}$$

equation of the tangent in a very simple form.

The 'm' of the first line $= -\dfrac{x_1}{y_1}$; and that of the second line $= \dfrac{y_1}{x_1}$.

The product of these two m's being $= -1$, the tangent is evidently perpendicular to the radius to the point of contact.[15]

18·3. To find the condition that the line

$$y = mx + c \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

may touch the circle $x^2 + y^2 = a^2$.

The equation of the tangent at any point (x_1, y_1) on the circle is

$$xx_1 + yy_1 = a^2. \quad \ldots \quad \ldots \quad (9)$$

If (8) touches the circle at (x_1, y_1), then (8) and (9) must represent the same straight line.[16] Hence,

$$\frac{x_1}{-m} = \frac{y_1}{1} = \frac{a^2}{c}.$$

$$\therefore \left.\begin{aligned} x_1 &= -\frac{a^2 m}{c} \\ \text{and } y_1 &= \frac{a^2}{c}. \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (10)$$

And since (x_1, y_1) is on the circle, we have $x_1^2 + y_1^2 = a^2$.

$$\therefore \frac{a^4 m^2}{c^2} + \frac{a^4}{c^2} = a^2, \text{ by (10).}$$

---

15 It may be advantageous for the student to note that the analytical proof given above is really a *verification* of the truth of the theorem. *This procedure will be followed in proving geometrical theorems throughout the book.*

16 Since a straight line cuts a circle (or any curve of the second degree) in two points, there is only one tangent at a point on the circle (or any curve of the second degree).

And since $a \neq 0$, we have

$$c^2 = a^2(1+m^2).$$

This is, therefore, the required condition.[17-19]

18·4. On the number of tangents to a circle from a given point outside it.

Let the equation of the circle be $x^2+y^2=a^2$; and let $P \equiv (\alpha, \beta)$ be the given point. The equation of any tangent to the circle is

$$y = mx + a\sqrt{1+m^2}.$$

If it passes through P, we have

$$\beta = m\alpha + a\sqrt{1+m^2}.$$

$$\therefore \qquad (\beta - m\alpha)^2 = a^2(1+m^2),$$

or
$$m^2(\alpha^2 - a^2) - 2m\alpha\beta - a^2 + \beta^2 = 0. \qquad \ldots \qquad (11)$$

As this is a quadratic equation in m, it is clear that there are two tangents (real or imaginary) which pass through any given point.

From (1), it is clear that these tangents will be real, coincident, or imaginary according as

$$4\alpha^2\beta^2 - 4(\alpha^2 - a^2)(\beta^2 - a^2) >, =, \text{ or } < 0;$$

i.e., according as $\beta^2 + \alpha^2 >, =, \text{ or } < a^2$.

---

[17] Note that the required condition has been obtained by eliminating x_1, y_1 from (10) and $x_1^2+y_1^2=a^2$.

It must be a relation between the constants occurring in the equation of the circle and that of the straight line, and must *not* contain x_1, y_1 which have been introduced for solving the problem.

[18] The condition of tangency could have been obtained from the fact that the length of the perpendicular from the centre of the circle to a given straight line is equal to the radius of the circle when, and and only when, the straight line touches the circle. This method depends upon a special property of the circle, and is not likely to be useful in case of other equations of the second degree.

[19] From the above condition, we have $c = \pm a\sqrt{1+m^2}$. Hence, the straight lines $y = mx \pm a\sqrt{1+m^2}$ are both tangents to the circle $x^2+y^2=a^2$ for all values of m.

But $\beta^2+\alpha^2 >, =,$ or $<a^2$ according as the point is without, on, or within the circle (Ex. 10, Art. 17). Hence, *the tangents will be real, coincident, or imaginary according as the given point is without, on, or within the given circle.*

18·5. To find the equation of the pair of tangents to the circle

$$x^2+y^2=a^2 \quad \ldots \quad \ldots \quad (12)$$

from a given point $P \equiv (x_1, y_1)$ outside it.

The equation of a line through $P \equiv (x_1, y_1)$ is

$$\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r, \quad \ldots \quad \ldots \quad (13)$$

where r is the algebraic distance of any point $Q \equiv (x, y)$ on it from (x_1, y_1), and θ = the angle which the upward direction of PQ makes with the axis of x.

The point Q is given by

$$x=x_1+r\cos\theta \text{ and } y=y_1+r\sin\theta.$$

If it lies on the circle, we must have

$$(x_1+r\cos\theta)^2+(y_1+r\sin\theta)^2=a^2.$$

$$\therefore\ r^2+2r(x_1\cos\theta+y_1\sin\theta)+x_1^2+y_1^2-a^2=0. \ldots \quad (14)$$

Now, if the straight line touches the circle at a point different from P, the roots of this equation in r must be equal and different from zero.[20] The condition for this is

$$(x_1\cos\theta+y_1\sin\theta)^2=x_1^2+y_1^2-a^2. \quad \ldots \quad \ldots \quad (15)$$

Eliminating θ from (13) and (15), we have

$$[x_1(x-x_1)+y_1(y-y_1)]^2=r^2(x_1^2+y_1^2-a^2), \quad \ldots \quad (16)$$

where $r^2=(x-x_1)^2+(y-y_1)^2$.

---

[20] A consideration of fig. 27 will tell us that this is so (see footnote 10, Art. 18·2).

As this is a relation between the co-ordinates of any point on either tangent from P to the circle, and as it does not contain any other variable quantity, it must be the equation of the pair of tangents.

To put it in another form, let

$$S \equiv x^2 + y^2 - a^2,$$
$$S_1 \equiv x_1{}^2 + y_1{}^2 - a_1{}^2$$
$$\text{and } T \equiv xx_1 + yy_1 - a^2.$$ [21]

Then, (16) becomes $(T - S_1)^2 = S_1(S + S_1 - 2T)$.

$$\therefore\ SS_1 = T^2,$$

a form in which the result can be easily remembered.

18·51. On the length of the tangent to a circle from a given point outside it.

(*i*) *When the equation of the circle is* $x^2 + y^2 = a^2$.

If $P \equiv (x_1, y_1)$ be the given point, then the product of the roots of r in (14)

$$= x_1{}^2 + y_1{}^2 - a^2. \quad \ldots \quad \ldots \quad (17)$$

= a quantity independent of the direction of the line through P.

For this reason, the square of the length of either tangent from P to the given circle is also

$$= x_1{}^2 + y_1{}^2 - a^2. \quad \ldots \quad \ldots \quad (18)$$

= a quantity which is *real* even if P is such that real tangents cannot be drawn from it. [22-23]

---

[21] Note that $S=0$ is the equation of the circle, S_1 is what S becomes when the co-ordinates of P are substituted in it, and $T=0$ is of the form of the equation of the tangent at $P \equiv (x_1, y_1)$ on the circle.

[22] It is not that the length of the tangent is *real* even when the tangent is *imaginary*. It is the square of it that is *real*.

[23] From what we have said above, it is clear that the rectangle contained by the segments of any chord of a circle drawn through a given point is constant, and is equal to the square of the tangent drawn from that point to the circle.

(ii) *When the equation of the circle is*

$$x^2+y^2+2fy+2gx+c=0. \quad \dots \quad \dots \quad (19)$$

To find where the line $\dfrac{x-x_1}{\cos\theta}=\dfrac{y-y_1}{\sin\theta}=r$ through P cuts (19), we have the following quadratic equation in r

$$r^2+2r[(x_1+g)\cos\theta+(y_1+f)\sin\theta]+x_1{}^2+y_1{}^2+2gx_1+2fy_1+c=0.$$

The product of the roots of this equation in r

$=x_1{}^2+y_1{}^2+2gx_1+2fy_1+c.$

= a quantity independent of the direction of the line through P.

For this reason, the square of the tangent from P to (19) is also

$$=x_1{}^2+y_1{}^2+2gx_1+2fy_1+c \quad \dots \quad \dots \quad (20)$$

= a quantity which is real for all positions of P.

[From (20), it is clear that the expression for the square of the tangent from a given point P to a given circle $S=0$ (which is so written that each of the coefficients of x^2 and y^2 is equal to unity) is obtained by substituting the co-ordinates of P in S.][24-25]

---

[24] The results of this article could have been proved by using the fact that the triangle formed by the point P, the point R which is the point of contact of the tangent from P, and the centre C of the circle is right-angled at R. But in order that a theorem of Elementary Pure Geometry may be applied, it is necessary that the ΔPRC should be real. Hence, this proof is restricted to the case when P is *without* the circle.

[25] It may be *noted* in this connection that the conception underlying imaginary points, lines, etc., is purely an algebraic one. They have no geometrical meaning as they cannot be located in a figure as real points, lines, etc. Hence the restriction imposed upon P in foot-note 24.

Examples.

1. Find the equation of the tangent to the curve $y^2 = 4ax$ at the point (x_1, y_1) on it.

[*Solution.* The equation of a line through $P \equiv (x_1, y_1)$ is

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r, \quad \ldots \quad \ldots \quad (21)$$

where r is the algebraic distance of any point $Q \equiv (x, y)$ on it from P, and θ = the angle which the upward direction of P makes with the axis of x. The point Q is given by

$$x = x_1 + r\cos\theta \text{ and } y = y_1 + r\sin\theta.$$

If it lies on the given curve, we must have

$$(y_1 + r\sin\theta)^2 = 4a(x_1 + r\cos\theta).$$

$$\therefore r^2\sin^2\theta + 2r(y_1\sin\theta - 2a\cos\theta) + y_1^2 - 4ax_1 = 0 \ldots (22)$$

The roots of this quadratic equation in r give the distances from P of the points of intersection of the line with $y^2 = 4ax$.

Now, if the straight line touches the curve $y^2 = 4ax$ at P, then both the roots of r in (22) must be equal to zero. The conditions for this are

$$y_1^2 - 4ax_1 = 0 \quad \ldots \quad \ldots \quad (23)$$

$$\text{and } y_1\sin\theta - 2a\cos\theta = 0. \quad \ldots \quad \ldots \quad (24)$$

Eliminating θ from (21) and (24), we have

$$\frac{x - x_1}{y - y_1} = \frac{y_1}{2a}, \text{ each side being equal to } \cot\theta.$$

$$\therefore \quad yy_1 - y_1^2 = 2a(x - x_1).$$

$$\text{or} \quad yy_1 = 2a(x + x_1), \text{ by (23).}$$

This is the equation of the tangent in its simplified form.] [26]

[26] Note that the process followed here is the same (step by step) as that of Art. 18·2 (ii).

**2. Find the equation of the tangent to the circle

$$x^2+y^2+2fy+2gx+c=0, \qquad \ldots \quad (25)$$

at any point $P\equiv(x_1, y_1)$ on it.

[*Solution.* To find where the line $\dfrac{x-x_1}{\cos\theta}=\dfrac{y-y_1}{\sin\theta}=r$ through $P\equiv(x_1, y_1)$ cuts (25), we have the following equation in r

$$r^2+2r[(x_1+g)\cos\theta+(y_1+f)\sin\theta]+x_1^2+y_1^2+2fy_1 +2gx_1+c=0.$$

Now, if the line touches the circle at P, both the roots of the equation in r must be zero. The conditions for this are

$$(x_1+g)\cos\theta+(y_1+f)\sin\theta=0 \qquad \ldots \quad (26)$$

and $\quad x_1^2+y_1^2+2fy_1+2gx_1+c=0. \qquad \ldots \quad (27)$

Eliminating θ from (26) and the equations of the line, we have

$\dfrac{x-x_1}{y-y_1}=-\dfrac{y_1+f}{x_1+g}$, each side being equal to $\cot\theta$.

$\therefore \quad (x-x_1)(x_1+g)+(y-y_1)(y_1+f)=0,$

or $\quad xx_1+yy_1+gx+fy=x_1^2+y_1^2+gx_1+fy_1$

$$=-(gx_1+fy_1+c), \text{ by (27)}.$$

Hence, the equation of the tangent is

$$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0.]^{27}$$

3. Find the equation of the tangent at any point P on a curve in the following cases :—

---

[27] The method of Differential Calculus can, of course, be used in obtaining the equation of the tangent in this as well as in the previous case. This was not done to help the student in getting a mastery over the analysis necessary in such cases.

(i) $x^2+y^2-2ax=0.$

** (ii) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$

** (iii) $ax^2+2hxy+by^2+2fy+2gx+c=0.$[28]

4. Write down the equation of the tangent at (3,4) on the circle $x^2+y^2=25.$

5. Write down the equation of the tangent at the point 'θ' on the circle $x^2+y^2=a^2$.

**6. Find the condition that the straight line $y=mx+n$ may touch the circle $x^2+y^2+2gx+2fy+c=0.$

[*Hints.* If the given line touches the circle at (x_1, y_1), it

must be the same as $xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$, which is the equation of the tangent at (x_1, y_1). Hence,

$$\frac{x_1+g}{m}=\frac{y_1+f}{-1}=\frac{gx_1+fy_1+c}{n}=-\lambda\text{(say)}.$$

$\therefore \quad x_1+0.\,y_1+g+m\lambda=0,$

$\quad 0.x_1+y_1+f-\lambda=0.$

and $\quad gx_1+fy_1+c+n\lambda=0.$

Also $mx_1-y_1+n+0.\lambda=0$, since (x_1, y_1) lies on the given line.

Eliminating x_1, y_1 and λ from these equations, we have

$$\begin{vmatrix} 1, & 0, & g, & m \\ 0, & 1, & f, & -1 \\ g, & f, & c, & n \\ m, & -1, & n, & 0 \end{vmatrix} = 0$$ as the condition required.][29]

[28] The form of the equation of the tangent at any point $P\equiv(x_1, y_1)$ on a curve represented by an equation of the second degree in x and y shows that it can be obtained mechanically by writing xx_1 for x^2, yy_1 for y^2, xy_1+x_1y for $2xy$, $x+x_1$ for $2x$ and $y+y_1$ for $2y$ in the equation of the curve.

[29] This method of elimination (with the help of determinants) is very useful in a lengthy case like this. The ordinary method of

7. Find the condition that the straight line $x \cos \alpha + y \sin \alpha - p = 0$ may touch the circle $x^2 + y^2 - 2ax = 0$.

**8. Find the condition that the straight line $y = mx + c$ may touch a curve in the following cases :—

(i) $y^2 = 4ax$; (ii) $ax^2 + by^2 = 1$.

**9. If $lx + my + n = 0$ be a tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, it will also be a tangent to the circle $x^2 + y^2 + 2gx + 2fy + c + \lambda\ (lx + my + n) = 0$ for all values of λ.

10. Show that the straight line $y = x + a\sqrt{2}$ touches the circle $x^2 + y^2 = a^2$; and find its point of contact.

[*Hints.* Proceed as in Art. 183. Equations corresponding to (10) of this article give the point of contact. If the conditon of tangency is identically satisfied, the given straight line will touch the circle.]

11. Show that the straight line $x + y - 1 = 0$ touches the curve $x^2 + y = x$.

12. Find the condition that the straight line $a \cos \theta + b \sin \theta = 0$ may touch the circle $r = 2c \cos \theta$.

[*Hints.* Change to the Cartesian system, and proceed as usual.]

13. Find the equation of the pair of tangents to the curve $y^2 = 4ax$ from the point $P \equiv (x_1, y_1)$ outside it. [30]

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solving and substituting is generally convenient in simpler cases (*cf.* Art. 183).

[30] That two tangents can be drawn to $y^2 = 4ax$ from a given point outside it can be seen as follows :—

The tangent at a point (x', y') on the curve is $yy' = 2a\ (x + x')$. If it passes through $P \equiv (x_1, y_1)$, we must have $y_1 y' = 2a\ (x_1 + x')$. But since (x', y') is on the curve, we have $y'^2 = 4ax'$. These two

[*Solution.* To find where the line $\dfrac{x-x_1}{\cos\theta}=\dfrac{y-y_1}{\sin\theta}=r$ through $P\equiv(x_1, y_1)$ cuts the curve, we have the following quadratic equation in r

$$r^2\sin^2\theta+2r(y_1\sin\theta-2a\cos\theta)+y_1{}^2-4ax_1=0.$$

Now, if the line touches the curve at any point different from P, then the roots of this equation in r must be equal and different from zero. The condition for this is

$$(y_1\sin\theta-2a\cos\theta)^2=\sin^2\theta(y_1{}^2-4ax_1).$$

Eliminating θ from this and the equation of the line, we have

$$\{y_1(y-y_1)-2a(x-x_1)\}^2=(y-y_1)^2(y_1{}^2-4ax_1) \quad \ldots \; 28$$

as the equation of the pair of tangents.

To simplify it, let $S\equiv y^2-4ax$, $S_1\equiv y_1{}^2-4ax_1$ and $T\equiv yy_1-2a(x+x_1)$.

Then, (28) becomes $(T-S_1)^2=(S+S_1-2T)S_1$;

$$\therefore \quad SS_1=T^2.\text{]}[31]$$

**14. Find the equation of the pair of tangents to the circle $x^2+y^2-2ax=0$ from the point $P\equiv(x_1, y_1)$ outside it.

**15. Find the equation of the pair of tangents to the curve $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ from the point $P\equiv(x_1, y_1)$ outside it.

16. Find the length of the tangent to the circle $2y^2+2y^2=5$ from the point $(2, -3)$.

---

equations in x', y' have two solutions in common. Hence the result. [Proceeding similarly, it can be easily seen that the above result is true for any curve of the second degree.]

[31] As in foot-note 21, $S=0$ is the equation of the curve $y^2-4ax=0$, S_1 is what S becomes when the co-ordinates of P are substituted in it, and $T=0$ is of the form of the equation of the tangent at (x_1, y_1) on the curve $S=0$.

17. Find the length of the tangent to the circle $x^2 + y^2 - ax - by + c = 0$ from the point (a, b).

18. Find the equation of the circle which has (3, 4) as its centre and touches the straight line $3x + 7y = 1$.

**19. Find the general equation of the circle with reference to two perpendicular tangents as axes of co-ordinates.

ON CHORDS.

19·1. To find the equation of the chord joining the points (x_1, y_1) and (x_2, y_2) on the circle $x^2 + y^2 = a^2$.

Consider the equation

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = x^2 + y^2 - a^2. \quad \ldots \quad (1)$$

The terms of the second degree on one side of the equation destroy those on the other side of it. Hence, the equation represents a straight line. Again, by writing x_1 for x and y_1 for y in (1), we see that the left-hand side vanishes identically, and that the right-hand side vanishes on account of the fact that the point is on the circle. Hence, (1) passes through the point (x_1, y_1). Similarly, it can be shown that (1) passes through (x_1, y_1). Hence, (1) is the equation of the chord through (x_1, y_1) and (x_2, y_2).[32]

19·2. To find the equation of a chord of the circle $x^2 + y^2 = a^2$ in terms of the co-ordinates of its middle point.

[32] This method of finding the equation of a straight line joining two points on a curve is rather of an abrupt nature (*cf.* Art. 14·6). It is due to Burnside.

The equation of a line through $P \equiv (x_1, y_1)$

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r, \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

where r is the algebraic distance of any point $Q \equiv (x, y)$ on it from P, and θ is the angle which the upward direction of PQ makes with the axis of x.

The point Q is given by

$$x = x_1 + r\cos\theta \text{ and } y = y_1 + r\sin\theta.$$

If it lies on the circle, we must have

$$(x_1 + r\cos\theta)^2 + (y_1 + r\sin\theta)^2 = a^2.$$

$$\therefore \quad r^2 + 2r(x_1\cos\theta + y_1\sin\theta) + x_1^2 + y_1^2 - a^2 = 0.$$

Now, if $P \equiv (x_1, y_1)$ be the middle point of the chord whose equation is (2), the roots of this equation in r must be equal and opposite. The condition for this is

$$x_1\cos\theta + y_1\sin\theta = 0. \quad \ldots \quad \ldots \quad (3)$$

Eliminating θ from (2) and (3), we have

$$\frac{x - x_1}{y - y_1} = -\frac{y_1}{x_1}, \text{ each side being equal to } \cot\theta.$$

$$\therefore \quad x x_1 + y y_1 = x_1^2 + y_1^2,$$

which is the equation of the chord having (x_1, y_1) as its middle point.[33]

19·3. To find the equation of the chord of contact of tangents to the circle $x^2 + y^2 = a^2$ from a given point $P \equiv (x_1, y_1)$ outside it.

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[33] Note that the equation is of the form $T = S_1$ where T and S_1 have the same significance as in foot-note 21, Art. 18·5.

Let $Q \equiv (x', y')$ and $R \equiv (x'', y'')$ be the points of contact of the tangents from P to the circle.

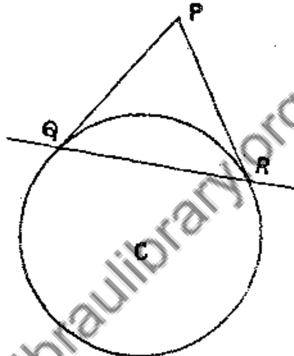

FIG. 28

The equation of the tangent at $Q \equiv (x', y')$ is $xx' + yy' = a^2$; and as it passes through $P \equiv (x_1, y_1)$, we must have

$$x_1 x' + y_1 y' = a^2. \quad \ldots \quad (4)$$

Similarly, since the tangent at $R \equiv (x'', y'')$ passes through $P \equiv (x_1, y_1)$, we have

$$x_1 x'' + y_1 y'' = a^2. \quad \ldots \quad (5)$$

From (4) and (5), it is clear that the points Q and R lie on the straight line whose equation is

$$xx_1 + yy_1 = a^2.$$

Hence, the equation of the chord of contact (*i. e.*, the line joining the points of contact) of tangents from (x_1, y_1) to the circle is

$$xx_1 + yy_1 = a^2.[34] \quad \ldots \quad \ldots \quad (6)$$

19·31. A geometrical theorem.

The chord of contact of tangents from a given point to a given circle is perpendicular to the line joining it to the centre of the circle.

Let the equation of the given circle be $x^2 + y^2 = a^2$; and let $P \equiv (x_1, y_1)$ be the given point.

The line joining $P \equiv (x_1, y_1)$ to the centre of the circle is

$$xy_1 - x_1 y = 0.$$

---

[34] This shows that *the line joining the points of contact is real even if the tangents from P are imaginary.*

And the chord of contact is $xx_1 + yy_1 = a^2$.

The product of their m's being $= -1$, the lines are evidently perpendicular to each other.

Examples.

1. Find the equation of the chord joining the points (x_1, y_1) and (x_2, y_2) on the curve $y^2 = 4ax$.

[*Solution*. Consider the equation $(y - y_1)(y - y_2) = y^2 - 4ax$. Since the terms of the second degree on one side of the equation destroy those on the other side of it, it represents a straight line. Again, it passes through (x_1, y_1) since its left-hand side vanishes identically if we write y_1 for y, and the right-hand side vanishes on account of the fact that the point (x_1, y_1) is on the curve $y^2 = 4ax$. Similarly, it can be shown that it passes through the point (x_2, y_2). Hence the result.]

**2. Show that the equation of the chord joining the points (x_1, y_1) and (x_2, y_2) on the curve $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$\frac{(x - x_1)(x - x_2)}{a^2} - \frac{(y - y_1)(y - y_2)}{b^2} = \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1.$$

*3. Show that the equation of the straight line joining the points 'θ' and 'ϕ' on the circle $x^2 + y^2 = a^2$ is

$$x \cos \frac{\theta + \phi}{2} + y \sin \frac{\theta + \phi}{2} = a \cos \frac{\theta - \phi}{2}.$$

**4. Find the equation of a chord of the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in terms of the co-ordinates of its middle point.

[*Solution.* Let the equation of the chord be

$$\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r. \quad \ldots \quad \ldots \quad (7)$$

Any point Q on this line whose algebraic distance from $P\equiv(x_1, y_1)$ is equal to r is $(x_1+r\cos\theta, y_1+r\sin\theta)$.

If it lies on $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we must have

$$\frac{(x_1+r\cos\theta)^2}{a^2}+\frac{(y_1+r\sin\theta)^2}{b^2}=1.$$

$$\therefore\ r^2\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right)+2r\left(\frac{x_1\cos\theta}{a^2}+\frac{y_1\sin\theta}{b^2}\right)$$

$$+\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1=0.$$

Now, if $P\equiv(x_1, y_1)$ be the middle point of the chord whose equation is (7), the roots of this equation in r must be equal and opposite. The condition for this is

$$\frac{x_1\cos\theta}{a^2}+\frac{y_1\sin\theta}{b^2}=0.$$

Eliminating θ from this and (7), we have

$$(x-x_1)\frac{x_1}{a^2}+(y-y_1)\frac{y_1}{b^2}=0,$$

which is the equation of the chord.][35]

**5. Show how to find the equation of a chord of a curve in terms of the co-ordinates of its middle point in the following cases :—

(i) $x^2+y^2-2ay=0$.

[35] *By Differential Calculus*, it can be easily seen the equation of a chord of the curve of the second degree whose equation is $f(x, y)=0$ in terms of the co-ordinates of its middle point $P\equiv(x_1, y_1)$ is $(x-x_1)\frac{df}{dx_1}+(y-y_1)\frac{df}{dy_1}=0$.

**(ii) $y^2 = 4ax$.

**(iii) $x^2 + y^2 + 2fy + 2gx + c = 0$.

6. Find the equation of the chord of contact of tangents to the curve $y^2 = 4ax$ from the point $P \equiv (x_1, y_1)$ outside it.

[*Solution.* Let $Q \equiv (x', y')$, $R \equiv (x'', y'')$ be the points of contact of tangents from P. Now, the equation of the tangent at Q is

$$yy' = 2a(x + x') \; ; \qquad \ldots \qquad \ldots \qquad (8)$$

and as it passes through $P \equiv (x_1, y_1)$, we must have

$$y_1 y' = 2a(x_1 + x').$$

Similarly, since the tangent at R passes through P, we have

$$y_1 y'' = 2a(x_1 + x''). \qquad \ldots \qquad \ldots \qquad (9)$$

From (8) and (9), it is clear that the points Q and R lie on the straight line whose equation is

$$yy_1 = 2a(x + x_1).$$

This is, therefore, the equation of the chord of contact.]

7. Find the equation of the chord of contact of tangents from a given point to a given curve in the following cases :—

(1) $x^2 + y^2 - 2ax = 0$.

**(2) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.

**(3) $x^2 + y^2 + fy + 2gx + c = 0$.

**(4) $ax^2 + by^2 + 2hxy + 2fy + 2gx + c = 0$.

8. Find the length of the chord joining the points in which the line $y = mx + c$ cuts the circle $x^2 + y^2 = a^2$.

[*Hints.* This can be done by finding the length of the perpendicular from the centre of the circle to the chord of contact, and then considering a suitable right-angled triangle.]

9. Find the length of the chord of contact of tangents from (3, 4) to the circle $x^2+y^2=7$.

ON NORMALS.

20·1. Definition.

By the normal to a circle (or a curve) at any point (x_1, y_1) on it is meant the straight line which passes through it and is perpendicular to the tangent to the circle (or the curve) at that point.

20·2. To find the equation of the normal to the circle $x^2+y^2=a^2$ at any point (x_1, y_1) on it.

The equation of any line through (x_1, y_1) is

$$y-y_1=m(x-x_1).$$

If it is perpendicular to the tangent at (x_1, y_1) whose equation is $xx_1+yy_1=a^2$, we must have.

$$m\left(-\frac{x_1}{y_1}\right)=-1.$$

$$\therefore \quad m=\frac{y_1}{x_1}.$$

Hence, the equation of the normal is

$$y-y_1=\frac{y_1}{x_1}(x-x_1),$$

$$\text{or} \quad xy_1-x_1y=0. \quad \ldots \quad (1)$$

[It is a straight line passing through the centre of the circle].[36]

[36] That the normal at any point on a circle is the line joining that point to the centre of the circle is clear from the result of Art. 18·21.

Examples.

1. Find the equation of the normal at any point (x_1, y_1) on the curve $y^2 = 4ax$.

[*Solution.* The equation of any line through (x_1, y_1) is

$$y - y_1 = m(x - x_1).$$

If it is perpendicular to the tangent at (x_1, y_1) whose equation is $yy_1 = 2a(x + x_1)$, we must have

$$m \times \frac{2a}{y_1} = -1.$$

$$\therefore \quad m = \frac{-y_1}{2a}.$$

Hence, the equation of the normal is

$$y - y_1 = -\frac{y_1}{2a}(x - x_1).]$$ [37]

2. Find the equation of the normal at any point (x_1, y_1) on a curve in the following cases :—

(1) $x^2 + y^2 - 2ax = 0.$

**(2) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1.$

**(3) $x^2 + y^2 + 2gx + 2fy + c = 0.$

3. Find the equation of the normal at the point 'θ' on the circle $x^2 + y^2 = a^2$.

---

[37] We have seen that the straight line $y - y_1 = m(x - x_1)$ is the tangent to the curve $f(x, y) = 0$ at the point (x_1, y_1) on it, if $m = -\dfrac{\delta f}{\delta x_1}\Big/\dfrac{\delta f}{\delta y_1}$. Hence, the equation of the normal at (x_1, y_1) is $(x - x_1)\Big/\dfrac{\delta f}{\delta x_1} = (y - y_1)\Big/\dfrac{\delta f}{\delta y_1}$.

4. To find the condition that the line $y = mx + c$ may be a normal to a curve in the following cases :—

(1) $x^2 + y^2 = a^2$.

(2) $x^2 + y^2 - 2x - 2y = 0$.

**(3) $y^2 = 4ax$.

21.

ON POLES AND POLARS.

21·1. Definitions.

By the *polar* of a point $P \equiv (x_1, y_1)$ with respect to a circle (or a curve of the second degree) is meant the locus of the points of intersection of tangents to the circle (or the curve) at the extremities of chords through P. The point P is called the *pole* of its polar.

21·2. To find the equation of the polar of the point $P \equiv (x, y)$ with respect to the circle $x^2 + y^2 = a^2$.

Let $Q \equiv (x_2, y_2)$ be any point on the polar of P. Then, by definition, the chord of contact of tangents from $Q \equiv (x_2, y_2)$ to the given circle must passes through $P \equiv (x_1, y_1)$.

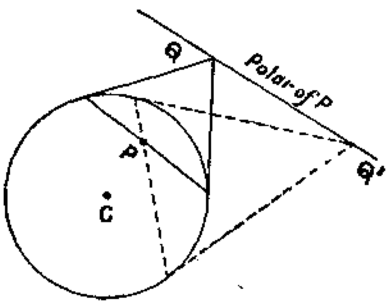

FIG. 29

Now, the equation of the chord of contact is $xx_2 + yy_2 = a^2$.

If it passes through $P \equiv (x_1, y_1)$, we must have

$$x_1 x_2 + y_1 y_2 = a^2.$$

But this latter equation shows that the point (x_2, y_2) must always lie on the straight line (*real*)

$$xx_1 + yy_1 = a^2. \qquad \ldots \qquad \ldots \qquad (1)$$

Hence, (1) is the equation of the *polar* of (x_1, y_1).[38]

21·21. Second definition of polar.

Comparing (1) with (6) of Art. 19·3, we see that the *polar* of a point P with respect to a given circle coincides with the chord of contact of tangents from P to the given circle. Hence, the polar of a point with respect to a given circle may be defined as the chord of contact of tangents from that point to the given circle. [This is, however, an *analytical* definition based on the notion of imaginary points.]

21·3. To find the pole of the line $y = mx + c$ with respect to the circle $x^2 + y^2 = a^2$.

Let $P \equiv (x_1, y_1)$ be the pole of the given line with respect to the given circle. Then, $y = mx + c$ must be the same as the polar of P with respect to the given circle.

Now, the equation of the polar of P is $xx_1 + yy_1 = a^2$.

Hence,
$$\frac{x_1}{-m} = \frac{y_1}{1} = \frac{a^2}{c}.$$

$$\therefore \qquad x_1 = -\frac{ma^2}{c} \text{ and } y_1 = \frac{a^2}{c}.$$

$\therefore$ the *pole* is at $\left(-\dfrac{ma^2}{c}, \dfrac{a^2}{c}\right)$.

---

[38] Note that the forms of the chord of contact of tangents from $P \equiv (x_1, y_1)$ and of the polar of $P \equiv (x_1, y_1)$ with respect to a circle (not passing through P) are exactly the same as that of the tangent at $P \equiv (x_1, y_1)$ on the assumption that P is on the circle.

21·31. To find the condition that the pole of $y=mx+c$ with respect to the circle $x^2+y^2=a^2$ may lie on $y=m'x+c'$.

By Art. 21·3, the pole of the 1st line is at $\left(-\frac{ma^2}{c}, \frac{a^2}{c}\right)$.

So, the condition that this point may lie on the 2nd line is

$$\frac{a^2}{c} = -\frac{mm'a^2}{c} + c'.$$

$$\therefore \quad a^2(1+mm') = cc',$$

which is the required condition.[30]

21·4. Some geometrical theorems.

(i) *If the polar of P with respect to a circle passes through Q, then the polar of Q will pass through P.*

Let the equation of the given circle be $x^2+y^2=a^2$; also let $P \equiv (x_1, y_1)$ and $Q \equiv (x_2, y_2)$ be any two given points.

Now, the polar of P is $xx_1+yy_1=a^2$.

If it passes through $Q \equiv (x_2, y_2)$, we must have

$$x_2x_1+y_2y_1=a^2. \quad \dots \quad (2)$$

Again, the polar of Q is $xx_2+yy_2=a^2$.

If it passes through P, we must have

$$x_1x_2+y_1y_2=a^2. \quad \dots \quad (3)$$

But the conditions (3) and (4) are the same. Hence,

---

[30] In the special case when $c=c'=0$, the lines become diameters of the circle; and the condition found above becomes $1+mm'=0$ (a being $\neq 0$). But this is the condition that the straight lines should be perpendicular to each other. Hence, the condition is identically satisfied by *any two perpendicular diameters of a circle.*

if one of them is satisfied, the other is automatically satisfied. Hence the theorem.[40–41]

(ii) *The line joining a point to the centre of a circle is perpendicular to the polar of the point with respect to the circle.*

Let the equation of the given circle be $x^2+y^2=a^2$; and let $P\equiv(x_1, y_1)$ be the given point.

Now, the polar of P is $xx_1+yy_1=a^2$; and the line joining P to the centre of the circle is $xy_1=yx_1$.

Hence, the 'm' of the first line is $-\dfrac{x_1}{y_1}$; and that of the second line is $\dfrac{y_1}{x_1}$. Their product being $=-1$, the lines are evidently perpendicular to each other. Hence the theorem.

21·5. Geometrical construction for the polar of a point.

Let the centre of the circle $x^2+y^2=a^2$ be at C; and let $P\equiv(x_1, y_1)$ be any point in its plane. Also let Q be the point where CP (or CP produced) is cut by the polar of P. Then, since CP is perpendicular to the polar of P, we have CQ = length of the perpendicular from C to the polar of P (fig. 30).

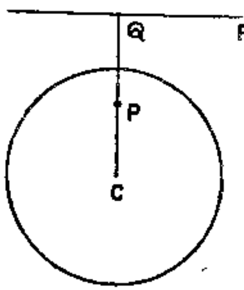

FIG. 30

[40] Two lines which are such that each passes through the *pole* of the other with respect to a given circle (or a curve of the second degree) are said to be *conjugate* to each other. [See foot-note 39]

[41] The proof of this theorem is also clear from the symmetry of the result of Art. 21·31.

$$\therefore \qquad CQ = \frac{a^2}{\sqrt{x_1^2 + y_1^2}}.$$

Also $\qquad CP = \sqrt{x_1^2 + y_1^2}.$

$\therefore \quad CP.\ CQ = a^2 =$ the square of the radius of the circle.

Hence, to construct the polar QR of a point P with respect to a circle having its centre at C and radius $= a$, we have only to join CP, take a point Q on CP (or CP produced) such that $OP.\ OQ = a^2$, and then draw QR perpendicular to CP.

Examples.

1. Find the polar of the point $P \equiv (x_1, y_1)$ with respect to the curve $y^2 = 4ax$.

[*Solution.* Let $Q \equiv (x_2, y_2)$ be any point on the polar of P. Then, by definition, the chord of contact of tangents from Q to the given curve must pass through P. Now, the equation of this chord is $y\,y_2 = 2a\,(x + x_2)$, by Ex. 6 of Art. 19. Hence, the condition that it may pass through P is

$$y_1 y_2 = 2a\,(x_1 + x_2).$$

This shows that the point $Q \equiv (x_2, y_2)$ must lie on the straight line $yy_1 = 2a\,(x + x_1)$.

This is, therefore, the equation of the polar of P.]

2. Find the equation of the polar of any point (x_1, y_1) with respect to a curve in the following cases :—

(1) $x^2 + y^2 - 2y = 0.$

** (2) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1.$

(3) $x^2 + y^2 - 2ay = 0.$

** (4) $x^2 + y^2 + 2gx + 2fy + c = 0.$

3. Write down the polar of the point $(-2, 3)$ with respect to the circle $x^2+y^2=5$.

4. Find the pole of the line $y=mx+c$ with respect to the curve $y^2=4ax$.

[*Solution.* Let $P\equiv(x_1, y_1)$ be the pole of the given line with respect to the given curve. Then, the line must be the same as the polar of P with respect to the given curve.

Now, the equation of the polar of P is $yy_1=2a(x+x_1)$.

Hence, $$\frac{2a}{m}=\frac{y_1}{1}=\frac{2ax_1}{c}.$$

$\therefore \quad x_1=\frac{c}{m}$ and $y_1=\frac{2a}{m}$.

$\therefore$ the pole is at $\left(\frac{c}{m}, \frac{2a}{m}\right)$.]

5. Find the pole of the straight line $2x-y=3$ with respect to the circle $x^2+y^2=5$.

6. Find the pole of the line $lx+my+n=0$ with respect to a curve in the following cases :—

(1) $x^2+y^2-2ax=0$.

**(2) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

**(3) $x^2+y^2+2gx+2fy+c=0$.

**(4) $ax^2+2hxy+by^2+2fy+2gx+c=0$.

7. *If the polar of P with respect to the curve $y^2=4ax$ passess through Q, then the polar of Q will pass through P.*

[*Solution.* The polar of $P\equiv(x_1, y_1)$ with respect to the curve $y^2=4ax$ is $yy_1=2a(x+x_1)$.

If it passes through $Q \equiv (x_2, y_2)$, we must have

$$y_2 y_1 = 2a(x_2 + x_1). \quad \dots \quad \dots \quad (5)$$

Again, the polar of Q is $yy_2 = 2a(x + x_2)$.

If it passes through P, we must have

$$y_1 y_2 = 2a(x_1 + x_2). \quad \dots \quad \dots \quad (6)$$

But the conditions (5) and (6) are identical. Hence the theorem.]

**8. If the polar of P with respect to the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ passes through Q, then the polar of Q will pass through P.

22.

ON SOME TOPICS CONNECTED WITH TWO OR MORE CIRCLES.

When two or more coplanar circles have to be dealt with in the same problem, it is advisable to refer them to a pair of rectangular axes similarly related (or unrelated) to each of them. [If the equations of the circles are written in *symmetrical* forms, the spade work connected with a given problem is likely to be easier than otherwise.]

22·1. To find the condition that two circles whose equations are given may cut each other orthogonally.[42]

Let the equations of the circles be

$$x^2 + y^2 + 2g_1 x + 2f_1 y + c_1 = 0,$$

[42] Two circles are said to cut each other *orthogonally* when the tangents at any point of intersection of these circles are at right angles.

and $x^2+y^2+2g_2x+2f_2y+c_2=0.$

Theire entres are at $C_1=(-g_1,-f_1)$ and $C_2=(-g_2,-f_2)$; and their radii are $\sqrt{g_1{}^2+f_1{}^2-c_1}$ and $\sqrt{g_2{}^2+f_2{}^2-c_2}$ respectively. Now, the tangent to a circle is perpendicular to the radius drawn to the point of contact.

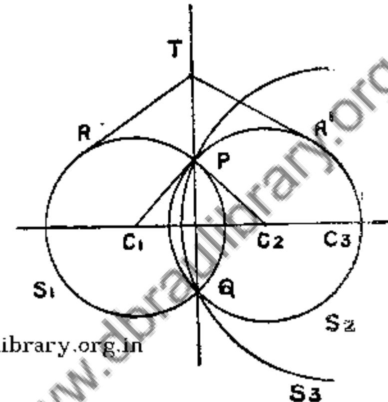

FIG. 31

Hence, if the tangents to the two circles at any point P common to them are at right angles to each other, the triangle C_1PC_2 must be a right-angled one (fig. 31). Hence,

$$C_1C_2{}^2=C_1P^2+C_2P^2.$$

$$\therefore \quad (g_1-g_2)^2+(f_1-f_2)^2=g_1{}^2+f_1{}^2-c_1+g_2{}^2+f_2{}^2-c_2,$$

or $\quad 2gg_1+2ff_1=c_1+c_2.$ ... ... (1)

This is, therefore, a *necessary* condition of orthogonality.

22·11. On the sufficiency of the condition (1).

To show that the necessary condition obtained above is also *sufficient* for the purpose, we write (1) as

$$(g_1-g_2)^2+(f_1-f_2)^2=g_1{}^2+f_1{}^2-c_1+g_2{}^2+f_2{}^2-c_2,$$

$\therefore \quad C_1C_2{}^2=C_1P^2+C_2P^2.$

$\therefore$ the angle C_1PC_2 is a right angle.

Hence, by Art. 18·21, (1) is a *sufficient* condition of orthogonality.

****22·2. On the locus of points from which the lengths of tangents drawn to two given circles are equal.**

Let the equations of the circles be

$$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0 \quad \ldots \quad (2)$$

and $S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0. \quad \ldots \quad (3)$

And let $T \equiv (x_1, y_1)$ be any point on the locus (fig. 31). Now, the square of the length of the tangent from T to (2) is $\quad x_1^2 + y_1^2 + 2g_1x_1 + 2f_1y_1 + c_1$;

and the square of the length of the tangent from P to (3) is $\quad x_1^2 + y_1^2 + 2g_2x_1 + 2f_2y_1 + c_2$.

Hence, by the condition of the problem, we have

$$x_1^2 + y_1^2 + 2g_1x_1 + 2f_1y_1 + c_1$$
$$= x_1^2 + y_1^2 + 2g_2x_1 + 2f_2y_1 + c_2.$$

$$\therefore \quad 2(g_1 - g_2)x_1 + 2(f_1 - f_2)y_1 + c_1 - c_2 = 0.$$

This is a relation between the co-ordinates (x_1, y_1) of any point T on the locus, and is *free* from any other unknown quantity. Hence, the equation of the locus is

$$2(g_1 - g_2)x + 2(f_1 - f_2)y + c_1 - c_2 = 0,$$

or $\quad S_1 - S_2 = 0. \quad \ldots \quad (4)$

It is a straight line, and is called the *radical axis* of the two circles.

****22·21. A geometrical theorem.**

The radical axis of two circles passes through the points of intersection (real or imaginary) of the circles, and is perpendicular to the line joining their centres.

Let the equations of the circles be

$$S_1 = 0 \text{ and } S_2 = 0,$$

where S_1 and S_2 are taken as in Art. 22·2 (with $+1$ as the co-efficient of x^2 in each case).

Now, the radical axis of the circles is given by

$$S_1 - S_2 = 0.$$

And this equation is satisfied whenever $S_1 = 0$ and $S_2 = 0$ are simultaneously satisfied (*i.e.*, at the points of intersection of $S_1 = 0$ and $S_2 = 0$).

Hence, the radical axis of two circles passes through their points of intersection (real or imaginary).[43-44]

Again, the m of the line $S_1 - S_2 = 0$ is $-\frac{g_1 - g_2}{f_1 - f_2}$; and that of the line joining the centres of the two circles is

$$\frac{f_1 - f_2}{g_1 - g_2}.$$

Their product being $= -1$, the line joining the centres is evidently perpendicular to the radical axis.

**** 22·22. To show that the radical axes of three circles taken in pairs meet in a point.**

Let the equations of the three circles be

$$S_1 = 0, \quad \ldots \quad \ldots \quad (5)$$
$$S_2 = 0 \quad \ldots \quad \ldots \quad (6)$$
$$\text{and } S_3 = 0. \quad \ldots \quad \ldots \quad (7)$$

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[43] In fig. 31, the radical axis of the circles $S_1 = 0$ and $S_2 = 0$ is represented by the line PQ. [The radical axis of two circles which touch each other is the common tangent at the point of contact.]

[44] Again, the radical axis $S_1 - S_2 = 0$ of the two circles cuts either $S_1 = 0$ or $S_2 = 0$ where $S_1 = 0$ and $S_2 = 0$ cut each other. But $S_1 - S_2 = 0$ cuts $S_1 = 0$ or $S_2 = 0$ in *two* points only (since one of the equations is of the first degree in x, y, and the other is of the second degree). This is contrary to the theory of equations which tells us that two curves, each of the second degree in x and y, intersect in four points (real or imaginary). To increase his outlook of things, the beginner may note that mathematicians have been able to explain this anomally by showing that the other two points of intersection are at infinity *i.e.*, at infinite distances from the centre of either circle).

(the equations being so written that the coefficient of x^2 may be unity in each case).

Now, the equations of the radical axes of (5) and (6), (6) and (7), and (7) and (5) *may* be written as

$$S_1 - S_2 = 0,$$
$$S_2 - S_3 = 0$$
$$\text{and } S_3 - S_1 = 0$$

respectively. And as the sum of the left-hand sides of these equations vanishes *identically*, it is clear that the three straight lines represented by them meet in a point.

[This point is called the *radical centre* of the three circles.]

** 22·23. A geometrical theorem.

If every pair of a system of circles have the same radical axis, the centres of these circles lie on a straight line.

Let the centres of the given circles be at C_1, C_2, C_3, etc. (fig. 31).

By Art. 22·21, we know that the radical axis of any two circles is perpendicular to the line joining the centres of these circles. Hence, the lines C_1C_2 and C_2C_3 through C_2 are both perpendicular to the same straight line (*viz.*, the common radical axis). But this is impossible unless C_1, C_2 and C_3 are collinear. Similarly, we can show that C_2, C_3 and C_4 are collinear; and so on. Hence, the centres C_1, C_2, C_3, C_4, etc. all lie in the same straight line perpendicular to the common radical axis.

22·3. To find the equation of a system of circles every pair of which has the same radical axis.[46]

[46] Such a system of circles is called a *coaxial system*.

By Art. 22·23, we know that the centres of these circles lie on the same straight line perpendicular to the common radical axis. So, if this line be taken as the axis of x, the ordinates of the centres of the circles must be zero. Hence, there can be no term in y in the equation of the circles.[47] Their equations must, therefore, be of the form

$$x^2 + y^2 + 2gx + c = 0. \quad \ldots (8)$$

Again, if the common radical axis be taken as the axis of y, the line $x = 0$ must cut the circles in the same two points. And since $x = 0$ cuts (8) in the two points whose ordinates are given by

$$y^2 + c = 0,$$

c must be the same for *all* such circles. Hence, the equation of this system of circles is given by (8), c being a known constant.

22·31. Notes on point-circles of the system.

(i) The equation $x^2 + y^2 + 2gx + c = 0$ can be written as $(x+g)^2 + y^2 = g^2 - c$.

This shows that the radius of any such circle will be zero, if $g^2 = c$. Hence, if $g = \pm\sqrt{c}$, the corresponding circles will degenerate into points coinciding with their centres.

The centres of these circles are $(\pm\sqrt{c}, 0)$, and are called the *limiting points* of the system of coaxial circles represented by (8).

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[47] If the equation of any circle of the system be $x^2 + y^2 + 2gx + 2fy + c = 0$, the centre of the circle is at $(-g, -f)$. Hence, if the ordinate of the centre is to be zero, f must be zero. Hence the result.

(ii) From the above, it is clear that *the limiting points are real or imaginary according as c is positive or negative.* Again, putting $x = 0$ in (8), we have

$$y^2 + c = 0.$$

Hence, the radical axis cuts *the circles in real or imaginary points according as c is negative or positive.* Combining these two results, we see that *the limiting points are real or imaginary according as the circles cut in imaginary or real points.*

***22·32. To find the equation of any circle coaxial with two given circles.**

Let the equations of the given circles be

$$S_1 = 0$$

$$\text{and } S_2 = 0,$$

where S_1 and S_2 are as in Art. 22·2.

Now, consider the equation $S_1 + \lambda S_2 = 0$, where λ is any constant.

Writing this out at full length, it can be easily seen that $S_1 + \lambda S_2 = 0$ satisfies the conditions for a circle. Hence, it represents a circle.

Again, the equation $S_1 + \lambda S_2 = 0$ is identically satsified whenever $S_1 = 0$ and $S_2 = 0$ are simultaneously true. Hence, it passes through the points of intersection of the two given circles for all values of λ. Hence, the radical axis of every pair of these circles is the same.

Hence, the equation of any circle coaxial with the two given circles is $S_1 + \lambda S_2 = 0$, where λ is any constant.

22·33. To find the limiting points of the coaxial system defined by two given circles.

Let the equation of the given circles be $S_1=0$ and $S_2=0$, where S_1 and S_2 are as in Art. 22·2.

Then, any circle of the coaxial system defined by the two given circles is of the form $S_1+\lambda S_2=0$, where λ is any constant. Its centre is at $[-(g_1+\lambda g_2), -(f_1+\lambda f_2)]$, and radius

$$=\{(g_1+\lambda g_2)^2+(f_1+\lambda f_2)^2-(c_1+\lambda c_2)\}^{\frac{1}{2}}.$$

Now, the limiting points are the centres of the circles of zero radius belonging to the system.

Hence, the limiting points are given by
$\{-(g_1+\lambda_1 g_2), -(f_1+\lambda_1 f_2)\}$ & $\{-(g_1+\lambda_2 g_2), -(f_1+\lambda_2 f_2)\}$, where λ_1 and λ_2 are the roots of λ in

$$(g_1+\lambda g_2)^2+(f_1+\lambda f_2)^2-(c_1+\lambda c_2)=0.$$

22·34. Some geometrical theorems.

(i) To show that the polar of a limiting point (of a system of coaxial circles) with respect to any circle of the system passes through the other.

Let the line of centres and the radical axis be taken as the axes of x and y respectively. Then, by Art. 22·3, the equation of any circle of the system is

$$x^2+y^2+2gx+c=0, \quad \ldots \quad (10)$$

where c is the same for all circles. The limiting points of this system are $A\equiv(\sqrt{c}, 0)$ and $B\equiv(-\sqrt{c}, 0)$.

$\therefore$ The polar of $A\equiv(\sqrt{c}, 0)$ with respect to (10) is $x\sqrt{c}+g(x+\sqrt{c})+c=0$.

$\therefore$ $(x+\sqrt{c})(g+\sqrt{c})=0$.

But, by hypothesis, $g\neq-\sqrt{c}$.

Hence, the equation of the polar of A is

$$x + \sqrt{c} = 0,$$

a straight line passing through the other limiting point $B \equiv (-\sqrt{c},\ 0)$. Hence the theorem.

[It is perpendicular to the *line of centres*.]

(ii) Every circle of a coaxial system is cut orthogonally by every circle passing through the limiting points.

Let the line of centres and the radical axis be taken as the axes of x and y respectively; and let the equation of any circle of the system be $x^2 + y^2 + 2gx + c = 0$, c being the same for all circles.

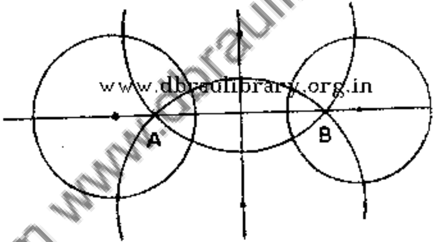

FIG. 32

The limiting points of the system are $A \equiv (\sqrt{c},\ 0)$ and $B \equiv (-\sqrt{c}, 0)$. Now, the centre of a circle passing through A and B must be on a line bisecting AB at right angles.

Hence, the abscissa of the centre of any circle passing through A and B must be zero (fig. 32).[48]

Hence, if the centre of any such circle be at $C \equiv (0, a)$, the equation of the circle is

$$x^2 + (y - a)^2 = a^2 + c,$$

its radius CA being $= \sqrt{a^2 + c}$.

$$\therefore \quad x^2 + y^2 - 2ay - c = 0.$$

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[48] The abscissa of the centre of any such circle must be the same as that of the middle point of AB.

Now, the condition that this circle may cut the given circle orthogonally is

$$2g.0 - 2a.0 - c + c = 0,$$

which is identically satisfied. Hence the theorem.

Examples.

1. Show that the circles $x^2 + y^2 + 2gx + c = 0$ and $x^2 + y^2 + 2fy - c = 0$ cut each other orthogonally.

2. Find the equation of the circle which passes through the origin and cuts the two circles $x^2 + y^2 - 4x + 6y + 10 = 0$ and $x^2 + y^2 + 2y + 6 = 0$ orthogonally.

[*Hints.* Let the equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

Now, express the conditions of the problem, determine the values of g, f and c from these conditions, and finally substitute their values in the above equation.]

3. Find the equation of the circle which cuts orthogonally the three circles

$$x^2 + y^2 = 4, \ (x - 3)^2 + y^2 = 4 \text{ and } x^2 + y^2 - 4y = 0.$$

[*Hints.* Proceed as in Ex. 2.]

4. *Find the equation of the circle cutting three given circles orthogonally.*

**5. Find the radical axis of the circles of Ex. 1.

**6. Find the radical centre of the three circles

$$x^2 + y^2 = 4, (x - 3)^2 + y^2 = 4 \text{ and } x^2 + y^2 - 4y = 0.$$

**7. Show that the difference of the squares of the lengths of the tangents to two given circles from any point in their plane varies as its distance from the radical axis of the circles.

**8. *If a circle cuts two given circles orthogonally, it will cut orthogonally all circles of the coaxial system defined by the given circles.*

**9. Find the limiting points of the coaxial system defined by the circles $x^2+y^2-2y+6=0$ and $x^2+y^2-4y+8=0$.

**10. Find the limiting points of the co ixial system defined by the circles $x^2+y^2+2x+6y+8=0$ and $x^2+y^2+6x+4x+10=0$.

**11. Show that the polars of any given point P with respect to a system of coaxial circles pass through a fixed point Q; and that PQ subtends a right angle at each of the limiting points of the system.

23.

ON SOME LOCUS PROBLEMS CONNECTED WITH THE CIRCLES

23·1. The direct method.

(I). *To find the locus of a point which moves so that the tangents from it to a given circle may be perpendicular to each other.*

Let the equation of the given circle be $x^2+y^2=a^2$; and let $P\equiv(x_1, y_1)$ be any point on the locus. Then, the equation of the pair of tangents from P to $x^2+y^2=a^2$ is

$$SS_1 = T^2,$$

or $$(x^2+y^2-a^2)(x_1{}^2+y_1{}^2-a^2)-(xx_1+yy_1-a^2)^2=0. \quad (1)$$

If these lines are perpendicular to each other, then the sum of the co-efficients of x^2 and y^2 in (1) must be zero.

$$\therefore \quad x_1{}^2+y_1{}^3-2a^2=0.$$

This is a relation between the co-ordinates of any point on the locus, and is *free* from any other unknown quantity. Hence, the equation of the locus is

$$x^2 + y^2 = 2a^2.$$

[It is a circle concentric with the given circle.]

(II) *To find the locus of the middle points of a system of parallel chords of a circle.*

Let the equation of the circle be $x^2 + y^2 = a^2$; and let $P \equiv (x_1, y_1)$ be any point on the locus. Now, the equation of the chord whose middle point is $P \equiv (x_1, y_1)$ is

$$xx_1 + y\,y_1 = x_1^2 + y_1^2.$$

If it is parallel to the line $y = mx + c$, then we must have

$$-\frac{x_1}{y_1} = m.$$

This is a relation between the co-ordinates of the middle point P of any chord of the system, and is *free* from any other unknown quantity. Hence, the locus of P is the straight line

$$x + my = 0.$$ [49]

(It is evidently perpendicular to the chords it bisects).

** 23·2. The parametric method.

To find the locus of the middle points of chords of a circle which subtend a right angle at a given point.

Let the equation of the given circle be $x^2 + y^2 = a^2$; and let $R \equiv (\alpha, \beta)$ be the given point. Also let $P \equiv (a \cos\theta, a \sin\theta)$, $Q \equiv (a \cos\phi, a \sin\phi)$ be any two

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[49] The locus need not be restricted to that portion of the straight line which is within the circle, as every straight line of the system cuts the circle in two points (real or imaginary).

points on it. Then, the coordinates (x, y) of the middle point of the chord PQ are given by

$$\left.\begin{aligned} x &= \frac{a}{2}(\cos\theta + \cos\phi), \\ y &= \frac{a}{2}(\sin\theta + \sin\phi). \end{aligned}\right\} \quad \ldots \quad (2)$$

Again, the 'm' of the line $PR = \dfrac{a\sin\theta - \beta}{a\cos\theta - \alpha}$;

and that of the line $QR = \dfrac{a\sin\phi - \beta}{a\cos\phi - \alpha}$.

$\therefore$ the condition of perpendicularity is

$$\frac{(a\sin\theta - \beta)(a\sin\phi - \beta)}{(a\cos\theta - \alpha)(a\cos\phi - \alpha)} = -1. \quad \ldots \quad (3)$$

Now, to obtain the equation of the locus, we must eliminate θ and ϕ from (2) and (3).

For this purpose, squaring and adding the equations (2), we have

$$x^2 + y^2 = \frac{a^2}{2}\left\{1 + \cos(\theta - \phi)\right\}. \quad \ldots \quad (4)$$

And re-arranging (3), we have

$$a^2\cos(\theta - \phi) - a\beta(\sin\theta + \sin\phi) - a\alpha(\cos\theta + \cos\phi) + \alpha^2 + \beta^2 = 0.$$

$\therefore$ $2(x^2 + y^2) - a^2 - 2\beta y - 2\alpha x + \alpha^2 + \beta^2 = 0$, by (2) and (4).

This is, therefore, the equation of the locus. [It is evidently a circle.] [50]

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[50] Unlike the locus problem (II) of Art. 23·1, a knowledge of *the co-ordinates of the end points of any chord of the system* has been made use of in this case. Hence, the difference of procedure in dealing with them (of course, from the point of view of expediency).

23·3. The method of polar co-ordinates.

(II) *Through a given point P any straight line is drawn cutting a given circle in Q_1 and Q_2, and a point Q is taken on it such that*

$$\frac{2}{PQ} = \frac{1}{PQ_1} + \frac{1}{PQ_2}; \text{ find the locus of } Q.$$

Let the equation of the given circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

with reference to any rectangular axes through the given point P as the *origin*. Then, the polar equation of the circle with reference to the origin as the *pole*, and the axis of x as the *initial line* is

$$r^2 + 2(g \cos\theta + f \sin\theta) r + c = 0.$$

$$\therefore \quad PQ_1 + PQ_2 = -2(g\cos\theta + f \sin\theta)$$

and $PQ_1 . PQ_2 = c$.

Now, since $\frac{2}{PQ} = \frac{1}{PQ_1} + \frac{1}{PQ_2} = \frac{PQ_1 + PQ_2}{PQ_1 . PQ_2}$.

$$\therefore \quad \frac{1}{r'} = -\frac{1}{c}(g\cos\theta + f\sin\theta), \text{ if } Q \equiv (r', \theta).$$

This is a relation between the polar co-ordinates of Q, and is *free* from any other unknown quantity. Hence, the polar equation of the locus of Q is given by

$$\frac{1}{r} = -\frac{1}{c}(g\cos\theta + f\sin\theta). \qquad \dots \qquad (5)$$

[That the locus is a straight line can be easily seen by changing (5) to Cartesian co-ordinates.]

**(II) *To find the locus of the foot of the perpendicular from a given point P to any chord of a circle which* *is such that it may subtend a right angle at the given point P.*

Let the equation of the circle be

$$x^2+y^2+2gx+2fy+c=0.$$

with reference to any rectangular axes through P; and let the equation of the chord be

$$x\cos\alpha+y\sin\alpha-p=0.\ [51]$$

Now, the equation of the pair of lines joining the origin to the points of intersection of the circle and the chord is

$$x^2+y^2+2(gx+fy)\left(\frac{x\cos\alpha+y\sin\alpha}{p}\right)+\frac{c}{p^2}$$
$$\times(x\cos\alpha+y\sin\alpha)^2=0.$$

If these lines are at right angles to each other, the sum of the co-efficients of x^2 and y^2 must be zero. Hence,

$$2p^2+2p(g\cos\alpha+f\sin\alpha)+c=0.$$

This is a relation between the polar co-ordinates (p, α) of the foot of the perpendicular from P on the chord, and is *free* from any other unknown quantity. Hence, the polar equation of the locus is

$$2r^2+2r(g\cos\theta+f\sin\theta)+c=0.\ [52]$$

[That the locus is a circle can be easily seen by changing its equation to one in Cartesian co-ordinates.]

[51] When it is required to find the locus of the foot of the perpendicular from a given point to a variable straight line, the equation of the line may conveniently be taken to be $x\cos\alpha+y\sin\alpha-p=0$ in its normal form.

[52] This is what the previous equation becomes when p and α are replaced by the current co-ordinates r and θ respectively.

*23·4. The elimination method.

A circle moves so that it may cut two given circles orthogonally; to find the locus of its centre.

Let the equation of the given circles be

$$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0 \quad \ldots \quad \ldots (6)$$

and $$S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0. \quad \ldots \quad \ldots (7)$$

And let the equation of the moving circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0. \quad \ldots \quad \ldots \quad \ldots (8)$$

Now, since (8) cuts (6) and (7) orthogonally, we must have

$$\left.\begin{array}{r} 2gg_1 + 2ff_1 - c - c_1 = 0 \\ \text{and } 2gg_2 + 2ff_2 - c - c_2 = 0. \end{array}\right\} \quad \ldots \quad \ldots \quad \ldots (9)$$

Again, the co-ordinates of the centre of the moving circle are given by

$$\left.\begin{array}{r} x = -g \\ \text{and } y = -f. \end{array}\right\} \quad \ldots \quad \ldots \quad \ldots (10)$$

The equation of the locus is, therefore, found by eliminating the variable parameters g, f and c from (9) and (10).

For this purpose, eliminating c from (9), we have

$$2g(g_1 - g_2) + 2f(f_1 - f_2) = c_1 - c_2.$$

$$\therefore \quad 2x(g_1 - g_2) + 2y(f_1 - f_2) = c_2 - c_1, \quad \ldots \quad \ldots (11)$$

by (10).

This is, therefore, the equation of the required locus.

23·41. Notes on the above.

The locus represented by (11) is evidently the *radical axis* of $S_1 = 0$ and $S_2 = 0$. Hence, the centre of a circle which cuts two given circles orthogonally lies on the radical axis of the given circles. Hence, the centre of the circle which cuts three given circles orthogonally is the same as the radical centre of the three circles (it being on the radical axis of any pair of them).

Again, if two circles cut each other orthogonally, the radius of any one of them is equal to the length of the tangent from its centre to the other circle. (This can be easily verified with the help of a diagram). Hence, the radius of a circle cutting three given circles orthogonally is equal to the length of the tangent from its centre to any one of the three circles.[53]

Examples.

1. Find the locus of points the tangents from which to the curve $y^2=4ax$ are at right angles to each other.

[*Solution.* Let $P\equiv(x_1, y_1)$ be any point on the locus. Then, the equation of the pair of tangents from P to the given curve is

$$SS_1=T^2,$$

or $(y^2-4ax)(y_1{}^2-4ax_1)-(yy_1-2ax-2ax_1)^2=0.$

If these lines are perpendicular to each other, the sum of the co-efficients of x^2 and y^2 in the above equation must be zero.

$$\therefore \quad 4a(x_1+a)=0.$$

But $a\neq 0$. Hence, the locus of P is given by $x+a=0$.]

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[53] These results can be used in finding the equation of the circle which cuts three given circles orthogonally (*cf.* Exs. 3 and 4 of Art. 22).

**2. Find the locus of points the tangents from which to the curve $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are at right angles to each other.

**3. Find the locus of the middle points of chords of the curve $y^2=4ax$ which pass through a given point.

4. Find the locus of a point which moves so that the length of the tangent drawn from it to a given circle may be $=k$ times the length of the tangent drawn from it to another fixed circle.

5. Find the locus of a point which moves in such a manner that the lengths of the tangents from it to two concentric circles are inversely as their radii.

6. Find the locus of points the polars of which with respect to two given circles make an angle of 45° with each other.

**7. Find the locus of the middle points of chords of $y^2=4ax$ which are parallel to a given line.

8. Find the locus of the middle points of chords of the circle $x^2+y^2=a^2$ which are of constant length.

9. Find the locus of the point of intersection of tangents to a circle at the extremities of chords of constant length.

[*Hints.* The tangents at $P\equiv(a\cos\theta, a\sin\theta)$ and $Q\equiv(a\cos\phi, a\sin\phi)$ on the circle $x^2+y^2=a^2$ are $x\cos\theta+y\sin\theta=a$ and $x\cos\phi+y\sin\phi=a$ respectively. If k be the length of the chord PQ, then $k^2=2a^2\{1-\cos(\theta-\phi)\}$. Eliminating θ and ϕ from these three equations, we have the equation of the required locus.]

10. Through a given point P any straight line is drawn cutting a given circle in Q_1 and Q_2. Find the locus of a point Q on it when

$$(1) \quad PQ_1 + PQ_2 = 2PQ.$$

$$(2) \quad \frac{1}{PQ^2} = \frac{1}{PQ_1^{\ 2}} + \frac{1}{PQ_2^{\ 2}}.$$

11. O is a fixed point and P any point on a given circle, and a point Q is taken on OP such that $OP.\,OQ = k^2$, find the locus of Q.[54]

12. O is a fixed point and P any point on a given circle. If a point Q is taken on OP such that $\frac{OQ}{OP} = k^2$ find the locus of Q.

13. Find the locus of the foot of the perpendicular from a fixed point on the circumference of a circle to any tangent to it.

14. Find the locus of a point which moves so that its polar with respect to a given circle may touch another fixed circle.

24.

MISCELLANEOUS TOPICS CONNECTED WITH CIRCLES.

24·1. *To find the length of the common chord of two circles whose equations are given.*

Let the equations of the circles be

$$x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$$

$$\text{and } x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0.$$

[54] A curve obtained in this manner from a given curve is said to be the *inverse* of the first curve with reference to O. [O is called the *centre of inversion*, and k the *radius of inversion*.]

Their centres are at $C_1 \equiv (-g_1, -f_1)$ and $C_2 \equiv (-g_2, -f_2)$; and their radii are $\sqrt{g_1^2 + f_1^2 - c_1}$ and $\sqrt{g_2^2 + f_2^2 - c_2}$ respectively.

Now, if PQ be the common chord and O its middle point, then C_1O is $\perp$ to PQ.

Hence, from the $\triangle\ C_1OP$, we have

$$PO^2 = C_1P^2 - C_1O^2.$$

$$\therefore \qquad PQ^2 = 4(C_1P^2 - C_1O^2). \quad \ldots \quad (1)$$

Substituting the values of C_1P and C_1O in (1), we have

$$PQ^2 = 4\left[g_1^2 + f_1^2 - c_1 - \frac{\{2g_1(g_1 - g_2) + 2f_1(f_1 - f_2) + c_2 - c_1\}^2}{4\{(g_1 - g_2)^2 + (f_1 - f_2)^2\}}\right],$$

since the equation of PQ is

$$2(g_1 - g_2)x + 2(f_1 - f_2)y + c_1 - c_2 = 0.$$

****24·2.** *To find the equations of the common tangents to two circles whose equations are given.*

Let C_1 and C_2 be the centres of the circles, and let their radii be a_1 and a_2 respectively. Also let a pair of common tangents meet the line C_1C_2 in T_1, and let the other pair meet it in T_2 (fig. 33).[55] Then, by the property of similar triangles, we have

$$\frac{C_1T_2}{C_2T_2} = \frac{a_1}{a_2} = \frac{C_1T_1}{T_1C_2}.$$

[55] That the points T_1 and T_2 are on the line joining C_1 and C_2 can be seen as follows:–

Let the common tangents P_1P_2 and Q_1Q_2 meet at T_1. Join C_1T_1 and C_2T_1. Now, we know that C_1T_1 bisects the angle $P_1T_1Q_1$, and C_2T_1 bisects the angle $P_2T_1Q_2$ which is equal to $P_1T_1Q_1$. But $P_1T_1Q_1 + Q_1T_1P_2 = 180°$. $\therefore$ $C_1T_1Q_1 + Q_1T_1C_2 = 180°$.

$\therefore$ C_1, T_1 and C_2 must be collinear. Similarly, it can be shown that C_1, T_2 and C_2 are collinear.

Hence, T_1 and T_2 divide C_1C_2 in the ratio of the radii of the circles.[56] Hence, the co-ordinates of T_1 and T_2

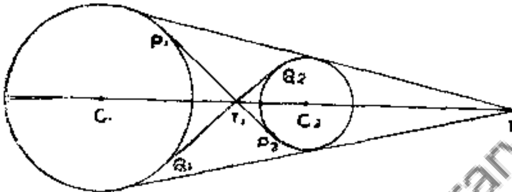

FIG. 33

can be found. Consequently, the equations of the four common tangents to the two circles are the same as the equations of the two pairs of tangents from T_1 and T_2 to either of the given circles. Hence, by Art. 18·5, the problem may be regarded as having been solved.

Examples.

**1. Find the length of the common chord of the circles $x^2+y^2+2x+5=0$ and $x^2+y^2+6y-5=0$.

**2. Find the length of the common chord of the circles $x^2+y^2+2x+17y+4=0$ and $x^2+y^2+5x+7y+25=0$.

**3. Find the equations of the common tangents of the circles

(i) $x^2+y^2=4$ and $(x-3)^2+y^2=25$.

(ii) $x^2+y^2-4x-6y+25=0$ and $x^2+y^2+4x-2y+4=0$.

---

[56] The points T_1 and T_2 which divide the line joining the centres of two circles in the ratio of their radii are called the *centres of similitude* for the two circles.

**4. Find the equation of the circle circumscribing the triangle formed by the lines $S_r \equiv x \cos \alpha_r + y \sin \alpha_r - p_r = 0$, $(r = 1, 2, 3)$ taken in order.

[*Hints.* Consider the equation $lS_1S_2 + mS_2S_3 + nS_3S_1 = 0$. It is an equation of the second degree in x, y passing through the vertices of the triangle. Now, write down the conditions under which it may represent a circle, and then use the equations thus obtained in showing that

$$\frac{l}{\sin(\alpha_2 - \alpha_1)} = \frac{m}{\sin(\alpha_3 - \alpha_2)} = \frac{n}{\sin(\alpha_1 - \alpha_3)}.$$ [57]

Hence the equation required.]

**5. Find the equation of the circle with respect to which a given triangle is self-conjugate. [58]

**6. Find the condition under which the quadrilateral formed by the lines $S_r \equiv x \cos \alpha_r + y \sin \alpha_r - p_r = 0$, $(r = 1, 2, 3, 4)$, taken in order may be inscribed in a circle.

[*Hints.* Consider the equation $S_1S_3 = \lambda S_2 S_4$.]

[57] By drawing a figure, it can be easily seen that $l : m : n :: \sin A : \sin B : \sin C$, A, B, C being vertices (properly chosen) of the triangle.

[58] A triangle is said to be *self-conjugate* with respect to a circle (or a curve of the second degree) when each vertex is the pole of the opposite side with regard to the given circle (or the curve of the second degree).

CHAPTER IV.

ON SOME POINTS CONNECTED WITH CONIC SECTIONS IN GENERAL.

25.

DEFINITIONS.

A conic section (or briefly, a *conic*) may be defined as the locus of a point P which moves in such a manner that its distance from a fixed point S bears a constant ratio to its distance from a fixed straight line XM.

The fixed point is called the *focus*, and the fixed line the *directrix*. The constant ratio (positive) is called the *eccentricity*, and is usually denoted by e. A conic section is called an *ellipse, parabola,* or *hyperbola* according as $e <, =,$ or $>$ unity.

26.

HISTORICAL NOTES.

The discovery of these curves is attributed to the Greek mathematician Manæchmus (about 400 B. C.) who noticed them in connection with the famous problem of the duplication of the cube (commonly known as the Delian problem). These curves were, however, first studied by Apollonius of Perga (about 250 B. C.) as plane sections of a double right circular cone; [1] and it is for this reason that the curves are called Conic Sections.

[1] A right circular cone is one in which the line joining its vertex to the centre of the circular base is perpendicular to the plane of the base.

The definition of a Conic Section as given in Art. 25 is based on a characteristic property of these curves known as the focus-directrix property. It was discovered by Pappus (about 400 A. D.)—nearly 800 years after they were first noticed by Manæchmus. It will be shown lateron (Art. 282) that any plane section of a cone having a circular base is a conic section as defined in Art. 25.

Another property that has been used in defining a *conic* is based on what may be regarded as an extension of the rectangular property of the circle. It states that $PN^2 = k.\ AN.NA'$ for an *ellipse* or *hyperbola*, and that $PN^2 = k.\ AN$ for a *parabola* (A, A' being the points of the curve lying on the line through S perpendicular to XM, N being the foot of the perpendicular from P on AA', and k being a constant). This property was discovered before the focus-directrix property, and has been used by Prof. Sommerville in defining a conic in his *Analytical Geometry of the Conic Sections.*

27.

CONIC SECTIONS AND THEIR EQUATIONS.

271. General form of the equation of a conic.

Let $S \equiv (x_1, y_1)$ be the focus, and $P \equiv (x, y)$ be any point on the conic. Also let the equation of the directrix XM be

$$x \cos \alpha + y \sin \alpha - p = 0.$$

Then, by definition, the equation of any conic whose focus is at S and directrix is XM is

$$\sqrt{(x-x_1)^2+(y-y_1)^2} = e\,(x\cos\alpha + y\sin\alpha - p),$$

or $$(x-x_1)^2 + (y-y_1)^2 = e^2(x\cos\alpha + y\sin\alpha - p)^2. \quad \ldots \quad (1)$$

[It is an equation of the second degree in x, y.] [2]

27·2. Every equation of the second degree in x, y represents a conic.

The most general equation of the second degree in x, y is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, \quad \ldots \quad \ldots \quad (2)$$

where a, h, b, g, f and c are arbitrary constants (none being *zero*).

Now, comparing (2) with (1), we have

$$\frac{1-e^2\cos^2\alpha}{a} = \frac{1-e^2\sin^2\alpha}{b} = \frac{-e^2\sin\alpha\cos\alpha}{h} = \text{etc.} \quad \ldots \quad (3)$$

And since we can always choose the five quantities x_1, y_1, e, α, p so as to satisfy the five equations (3) [or a lesser number of them], it is clear that (2) represents a conic (real or imaginary). [3] Moreover, it is easy to see that any equation of the second degree in x, y can be transformed into one of the type (2). Hence, *every equation of the second degree in x, y represents a conic* (real or imaginary).[4]

27·21. Some special cases of the conic section.

(i) Let $a = b$ and $h = 0$. Then, (2) becomes

$$a(x^2+y^2) + 2gx + 2fy + c = 0.$$

---

[2] That every equation of the form (1) represents a conic can be easily seen by proceeding backward from the above equation.

[3] In special cases, the equation (3) may reduce to less than five independent equations. But in that case, it is more convenient to choose x_1, y_1, e, α, p so that the equations (3) may be satisfied.

[4] As the degree of an equation in x, y is unaltered by any change of axes, it is clear that (2) represents a conic even when the axes are *oblique*.

$$\therefore \qquad \left(x+\frac{g}{a}\right)^2+\left(y+\frac{f}{a}\right)^2=\frac{g^2+f^2-ac}{a^2}.$$

Hence, it represents a circle (real or imaginary).

But, in the special case when $g^2+f^2=ac$, the circle degenerates into the point $\left(-\frac{g}{a}, -\frac{f}{a}\right)$.

Hence, *a circle (including a point) may be regarded as a special case of the conic section.*

(ii) Again, let the constants in (2) satisfy the relation

$$\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}=0.$$

Then, by Art. 9·41, the equation (2) represents a pair of straight lines (real or imaginary). Hence, *a pair of straight lines may also be regarded as a special case of the conic section.*

27·22. Note on the conditions necessary to determine a conic.

The most general equation of the second degree in x, y contains five independent constants [equation (2) being divisible throughout by any one of the constants a, b, c, f, g, h]. Hence, a conic may be made to satisfy *five* conditions which are such that each may give rise to *one* independent equation between the constants. Thus, a conic may be made to pass through five given points, or to pass through four given points and touch a given straight line; and so on.[5]

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[5] Any condition which gives rise to two or more independent equations between the constants must, for this purpose, be supposed to be equivalent to two or more conditions. Moreover, these condi-

**273. Some lemmas.

(*i*) Every straight line lying in the plane of a conic cuts it in two, and only two, points (*real or imaginary*).

A straight line is represented by an equation of the first degree in x, y; and a conic section by one of the second degree in x, y. Now, the two equations hold together at the points of intersection of the line and the conic, and at no other points. And since these equations admit of two and only two solutions (real or imaginary), it is clear that there are two and only two points (real or imaginary) common to a straight and a conic.

(*ii*) *If a curve is cut by any straight line lying in its plane) in two points (real or imaginary), it is a conic.*

A knowledge of the Theory of Equations tells us that two plane curves of the m^{th} and n^{th} degrees respectively intersect each other in mn points (real or imaginary). Hence, if two curves intersect in two points, and if one of them is a *straight line* (*i.e.*, a curve of the first degree), then the other must be a curve of the second degree (*i.e.*, a *conic*). Hence the theorem.

28.

CONIC SECTIONS AND PLANE SECTIONS OF A CONE.

28·1. Notes on projection and section.

If a point P be joined to a fixed point V, and VP (or VP produced) be cut by any fixed plane in P', then

tions may be such that more than one conic may be found to satisfy them; but when the five conditions are independent of each other, then the number of conics satisfying them will, by the theory of equations, be a definite one.

P' is called the *projection* of P on that plane. V is called the *centre of projection*, and the cutting plane is called the *plane of projection*.

If all the points of a closed plane curve A be joined to any fixed point V (not lying in the plane of the curve) and their projections on a second plane B are taken, then these points will lie on another curve C called the projection of the given curve on the plane B. This curve of projection may also be called the section by the plane B of the cone formed by joining V to all points of the curve A.

From what has been said above, it is clear that any plane section of a cone having a circular base may be regarded as the projection of the circle on the plane of the section.

From the definition, it is also clear that *the projection of any straight line on a given plane is always a straight line.* For, the straight lines joining V to all points of any given straight line are in one plane, and this is cut by the plane of projection in another straight line.

****28·2. Every plane section of a cone having a circular base is a conic.**

Corresponding to every point common to a straight line lying in the plane of the circular base and the circle, there must be a point common to the projections of the straight line and the circle on any plane. And as a straight line cuts a circle in two points (real or imaginary), there must be two points (real or imaginary) common to the projections of the straight line and the circle on any plane.

Now, the projection of a straight line on any plane is a straight line. Hence, by lemma (ii) of Art. 27·3, the projection of a circle on any plane must be a curve of the second degree (*i.e.*, a *conic*). And as this curve of projection may be regarded as the section of a cone having a circular base, the truth of the above theorem is established.[6]

Examples.

1. The co-ordinates x and y of any point P on a curve are given by

$$x = at^2 + bt,$$
$$\text{and } y = lt^3 + ct\,;$$

show that the locus of P, for different values of t, is a conic section

2. The co-ordinates x and y of any point P on a curve are given by

$$x = a \tan(\theta + \alpha),$$
$$y = b \tan(\theta + \beta)\,;$$

show that the locus of P, for different values of θ, is a conic.

3. Show that the curve whose polar equation is $r \sin\theta + \dfrac{1}{r\cos\theta} = r\cos\theta + 2$ is a conic section.

---

[6] We shall now make a detailed study of *the parabola*, *the ellipse* and *the hyperbola*.

CHAPTER V.

THE PARABOLA.

29.

ON THE PARABOLA AND ITS EQUATION.

29·1. Definition and general equation of a parabola.

For a parabola, $e=1$. Hence, a parabala is the locus of a point P which moves so that its distance from a fixed point S is always equal to its distance from a fixed straight line.[1]

Let $S\equiv(x_1, y_1)$ be the focus, and $P\equiv(x, y)$ be any point on the parabola; also let the equation of the directrix be $x \cos \alpha + y \sin \alpha - p = 0$.

Then, by definition, the equation of the parabola is

$$(x-x_1)^2+(y-y_1)^2=(x \cos \alpha + y \sin \alpha - p)^2 \quad \ldots \quad (1)$$

29·2. To find the standard form of the equation of the parabola.

Let S be the focus, and XM the directrix. Draw SX perpendicular to XM, and let $XS=2a$.

Also let XS and XM be taken as the axes of x and y respectively (fig. 34). Then, the focus S is $(2a, 0)$; and if $P\equiv(x, y)$ be any point on the parabola, its equation is

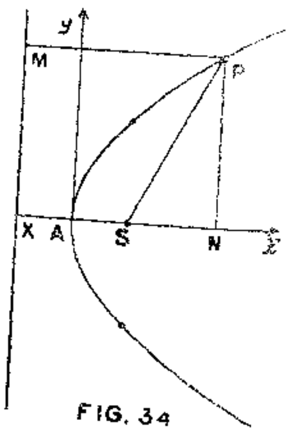

FIG. 34

[1] As in Art. 25, the fixed point is called the *focus*, and the fixed straight line is called the *directrix* of the parabola.

$$(x-2a)^2+y^2=x^2. \qquad \dots \qquad \dots \quad (2)$$

$$\therefore \qquad y^2=4a(x-a).$$

Changing the origin to $(a, 0)$ without changing the directions of the axes, we have

$$y^2=4ax \qquad \dots \qquad \dots \quad (3)$$

as the equation of the parabola in what is known as its *standard form*.

29·21. On the form of the curve whose equation is given by (3), the axes being rectangular.

Changing the origin to $(-a, 0)$ without changing the directions of the axes, we have

$$y^2=4a(x-a).$$

$$\therefore \qquad (x-2a)^2+y^2=x^2.$$ [2]

Hence, the distance of any point (x, y) on the curve from the fixed point $(2a, 0)$ = its distance from the line whose equation is $x=0$ (with reference to the new axes XS and XM). Hence, by definition, the curve is a *parabola*.

29·3. On some points connected with the parabola.

(i) Since the equation $y^2=4ax$ is unaltered by writing $-y$ for y, the curve is symmetrical about the axis of x. For this reason, the axis of x (which is the line through S perpendicular to the directrix) is called the *axis* of the parabola. (The curve is not symmetrical about the axis of y).

[2] This is equivalent to proceeding backwards from (3) until we arrive at (2).

If N be the foot of the perpendicular from any point P on the parabola to its axis, PN is called the *ordinate* of the point P The double-ordinate through the focus is called the *latus-rectum.*

(ii) Putting $y=0$ in $y^2=4ax$, we have $x=0$. Hence, the axis of the parabola cuts it at the origin. The point where a parabola is cut by its axis is called the *vertex* of the parabola.

Putting $x=0$, we have $y^2=0$. Hence, the line $x=0$ cuts the parabola in two coincident points (i.e., touches it) at the origin (the vertex).

(iii) As x increases, y increases too; and as there is no limit to this increase of x and y, the curve extends to infinity (of course, symmetrically about its axis).

As the left-hand side of $y^2=4ax$ is *positive*, its right-hand side must also be so. Hence, x cannot be *negative*; and the curve must lie wholly on the right-hand side of the axis of y (fig. 34).[3]

(iv) From Art. 29·2, it is clear that the *directrix* of the parabola whose equation is $y^2=4ax$ is given by $x+a=0$, and that its *focus* is at $(a, 0)$.

Putting $x=a$ in $y^2=4ax$, we have $y=\pm 2a$. Hence, the *latus-rectum* (i.e., the double-ordinate through the focus) $=4a$.

(v) Transferring the origin to $(h, 0)$ without changing the directions of the axes, we have

$$y^2=4a(x+h). \quad \dots \quad \dots \quad (4)$$

[3] In fig. 34, a is supposed to be *positive*. If it were negative, the curve would still be on *one side* (viz., left-hand side) of the y-axis.

Now, suppose that h remains *finite* while a tends to zero. Then, in the limit when $a=0$, (4) becomes $y^2=0$, which represents a pair of coincident straight lines.

Hence, *a pair of coincident straight lines may be regarded as a special case of the parabola.*

Again, suppose that ah remains *finite* while a tends to zero. Then, in the limit when $a=0$, (4) becomes $y^2=4ah$, a pair of parallel straight lines.

Hence, *a pair of parallel straight lines may also be regarded as a special case of the parabola.**

****29·4. To find the condition that the general equation of the second degree in x, y may represent a parabola.**

The most general equation of the second degree in x, y is

$$ax^2+2hxy+by^2+2gx+2fy+c=0, \quad \ldots \quad (5)$$

where a, h, b, g, f and c are arbitrary constants (none being *zero*).

Comparing (5) with (1), we have

$$\frac{a}{\sin^2\alpha}=\frac{h}{-\sin\alpha\cos\alpha}=\frac{b}{\cos^2\alpha}=\frac{f}{-x_1+p\cos\alpha}$$

$$=\frac{g}{-y_1+p\sin\alpha}=\frac{c}{x_1^2+y_1^2-p^2}.$$

* Besides giving other informations, the results of this article enable us to find the co-ordinates of the *vertex* and the *focus*, the equations of the *axis*, the *directrix* and the *tangent* at the vertex, and the length of the *latus-rectum* of the parabola (as represented by an equation of the second degree in x, y) by reducing its equation to the form $y^2=4ax$(the axes being rectangular). For illustrative examples, see the end of this section.

From the first two equations of this set, we have $ab - h^2 = 0$ as a *necessary* condition. [5]

29·41. The condition $ab - h^2 = 0$ is sufficient to make (5) represent a parabola.

By Art. 27·1, the equation of a conic can be written as

$$(x - x_1)^2 + (y - y_1)^2 = e^2(x \cos \alpha + y \sin \alpha - p)^2.$$

Comparing this with (5), we have

$$\frac{a}{1 - e^2 \cos^2 \alpha} = \frac{h}{-e^2 \sin \alpha \cos \alpha} = \frac{b}{1 - e^2 \sin^2 \alpha} = \text{etc.}$$

Now, by hypothesis, we have $ab - h^2 = 0$.

$$\therefore \quad (1 - e^2 \cos^2 \alpha)(1 - e^2 \sin^2 \alpha) - e^4 \sin^2 \alpha \cos^2 \alpha = 0;$$

$$\therefore \quad e^2 = 1, \quad \text{or } e = 1.$$

Hence, the conic represented by (5) is a *parabola*, if $ab - h^2 = 0$. [6] ... ... (6)

29·42. On the number of conditions necessary to determine a parabola.

The most general equation of the second degree in x, y contains five independent constants. But, since they are connected by one more relation, *viz.* $ab - h^2 = 0$, there are only *four* independent constants in the equation of a parabola. Hence, four conditions are *necessary and sufficient* to determine a parabola, these conditions being such that each may give rise to one equation between the constants. [7]

---

[5] When $ab - h^2 = 0$, *the terms of the second degree in* (5) *form a perfect square.* This can be easily seen by writing $\sqrt{ab}$ for h in (5).

[6] Hence, the equation of a parabola can always be written in the form $(\alpha x + \beta y)^2 + 2gx + 2fy + c = 0$.

[7] Any condition which gives rise to two (or more) equations between the constants must, for this purpose, be regarded as equivalent to two (or more) conditions.

****29·5. On the parabola as represented by the general equation of the second degree in *x, y*.**

If the most general equation of the second degree in x, y represents a parabola, its equation can be written as

$$(\alpha x+\beta y)^2+2gx+2fy+c=0.$$

This is the same as

$$(\alpha x+\beta y+k)^2=2(k\alpha-g)x+2(k\beta-f)y+k^2-c. \quad \ldots \quad (6)$$

Choosing k so that the straight lines

$$\alpha x+\beta y+k=0 \quad \ldots \quad \ldots \quad (7)$$

and $$2(k\alpha-g)x+2(k\beta-f)y+k^2-c=0. \quad \ldots \quad (8)$$

may be perpendicular to each other, we have

$$\alpha(k\alpha-g)+\beta(k\beta-f)=0.$$

$$\therefore \qquad k=\frac{\alpha g+\beta f}{\alpha^2+\beta^2}.$$

When k has this value, the equation of the *axis* is given by (7), and the equation of the *tangent at the vertex* is given by (8). The *vertex* is, therefore, the point of intersection of (7) and (8).

Again, (6) can be written as

$$\left(\frac{\alpha x+\beta y+k}{\sqrt{\alpha^2+\beta^2}}\right)^2 . (\alpha^2+\beta^2)$$

$$=\frac{2x(k\alpha-g)+2y(k\beta-f)+k^2-c}{2\sqrt{(k\alpha-g)^2+(k\beta-f)^2}} . \; 2\sqrt{(k\alpha-g)^2+(k\beta-f)^2}.$$

This can be written in the form

$$(-x\sin\theta+y\cos\theta-k)^2=l\,(x\cos\phi+y\sin\phi-h), \quad \ldots \quad (9)$$

where $l=\dfrac{2\sqrt{(k\alpha-g)^2+(k\beta-f)^2}}{\alpha^2+\beta^2}$

$$=\frac{2(\alpha f-\beta g)}{(\alpha^2+\beta^2)^{\frac{3}{2}}} \quad \ldots \quad \ldots \quad (10)$$

Hence, by Art. 4·3, (9) can be written as

$$y_1^2 = lx_1.$$

Hence, the length of the *latus-rectum* is equal to the absolute value of l, which is given by (10).

*29·6. The position in oblique co-ordinates.

Consider the equation $(\alpha x+\beta y)^2 = -(2gx+2fy+c)$, a form in which the equation of a parabola can always be written (the axes being *rectangular*). To transform it into one in any system of *oblique* axes is to replace x and y in the above equation by certain expressions of the first degree in the new co-ordinates. But this cannot change the *squared form* of its left-hand side, nor raise the degree of either side of it. Hence, *the terms of the second degree in x, y in the equation of a parabola form a perfect square even when the axes are oblique.*

29·7. Parametric equations of the parabola.

Putting $x=at^2$ in $y^2=4ax$, we have $y=2at$ as a solution. This shows that $(at^2, 2at)$ is a point on the parabola $y^2=4ax$ for *all* values of t (one point for each value of t). For this reason, the equations

$$\left.\begin{aligned} x &= at^2 \\ \text{and } y &= 2at \end{aligned}\right\} \qquad \ldots \qquad \ldots \quad (11)$$

in which the co-ordinates are expressed in terms of a parameter t are called the *parametric* equations of the parabola $y^2=4ax$.

[From the mode of procedure, it is clear that the equations (11) represent *only one* (though a very useful one) of the infinite number of ways in which the co-ordimates of any point on the parabola $y^2 = 4ax$ can be expressed in terms of a single parameter.]*–9

Examples.

1. Find the equation of the parabola whose focus is (2, 1) and directrix is the line $3x - 4y + 1 = 0$.

2. Find the equation of the parabola whose focus is the origin and directrix is the line $\frac{x}{3} + \frac{y}{4} = 1$.

**3. Find the equation of the parabola which touches the axes of co-ordinates (rectangular) at distances 3 and 4 from the origin.

**4. Show that $(3x + 4y - 1)^2 = 4x - 3y$ represents a parabola. Find its latus-rectum.

**5. *Find the latus-rectum, axis, vertex and focus of the parabola* $y^2 = 2x + 4y$.

[*Solution*. The equation can be written as $(y-2)^2 = 2(x+2)$. Now, changing the origin to the point $O_1 \equiv (-2, 2)$ without changing the directions of the axes, we have $y_1^2 = 2x_1$. Hence, its *latus-rectum* = 2. Again, with reference to the original axes, the *axis* O_1x_1 is the line $y = 2$ (fig. 35), the *vertex* is the point $O_1 \equiv (-2, 2)$ and the focus is the point $S \equiv (-\frac{3}{2}, 2)$.]

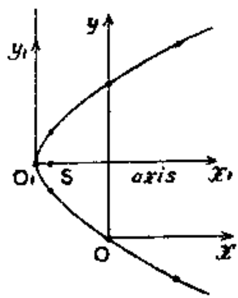

FIG. 35

* Given a conic by an equation of the second degree in x, y, we can always obtain two equations expressing x and y in terms of a single parameter. For, by putting $x = t$ (or any other convenient function of t) in the given equation, we can express y in terms

**6. Find the latus rectum, axis, vertex and focus of a parabola in the following cases :—

(1) $x^2 - 4ay = 0$.

(2) $y^2 = 4y - 4x$.

(3) $x^2 + 2xy + y^2 = 4x - 2y$.

**7. Trace the curve $y^2 = 2x + 4y$.

[*Hints.* (i) Since $ab - h^2 = 0$ in this case, the conic is a *parabola.* (ii) Its latus-rectum, axis, vertex and focus have been found in Ex. 5. (See Art. 29·5). (iii) The equation reduces to $y_1{}^2 = 2x_1$, with reference to parallel axes through O_1. (iv) Considering the original equation, we see that the parabola cuts the y-axis at the points $O \equiv (0, 0)$ and $P \equiv (0, 4)$; also it cuts the x-axis at the point $O \equiv (0, 0)$ only. (iv) Referred to the new axes, the points $(4, 2\sqrt{2})$, $(4, -2\sqrt{2})$, etc. are on the parabola (using $y_1{}^2 = 2x_1$ in this case). (v) Plot all these points carefully, and use your knowledge of the nature of the parabola in drawing a curve (fig. 35) through them. (See foot-note 64 of Ch. I).]

**8. Trace the parabolas :—

(1) $x^2 + 4ay = 0$.

(2) $(x + y)^2 = 2x + y + 1$.

(3) $4x^2 - 20xy + 25y^2 - 6x - 8y + 14 = 0$.

9. Show that $\left(\dfrac{\tan^2\theta}{a}, 2\tan\theta\right)$ is a point on a parabola for different values of θ.

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of t by solving the resulting equation in y (and taking only one of its values). These equations form a parametric set for the conic as the original equation can evidently be obtained by eliminating t from them.

[9] By proceeding exactly in the above manner, we can obtain as many sets of parametric equations (useful or useless) for the curve as we please.

10. Show that the equations $x = a_1t^2 + 2b_1t + c_1$ and $y = a_2t^2 + 2b_2t + c_2$, where t is a variable parameter, represent a parabola.

11. Show that the equations $x = 6 + t^2$ and $y = 2t$, where t is a variable parameter, represent a parabola; also find its axis and latus-rectum.

12. Show that the point $Q \equiv (x_1, y_1)$ lies *within*, *on*, or *without* the parabola $y^2 = 4ax$ according as $y_1^2 - 4ax_1$ is *negative*, *zero*, or *positive*.

[*Solution.* Let Q be within the parabola (*i. e.*, between the curve and its axis). Draw the ordinate NP, and let P be the nearer of the two points at which it meets the parabola. Then $QN^2 < PN^2 < 4ax_1$, since P is on the curve.

$\therefore$ $y_1^2 - 4ax_1$ is *negative*. Similarly, the other results can be proved.]

30.

ON TANGENTS.

By proceeding exactly as in the case of the circle (Art. 18·2), it can be easily seen that the equation of the tangent to the parabola $y^2 = 4ax$, at any point $P \equiv (x_1, y_1)$ on the curve is $yy_1 = 2a(x + x_1)$.[10] Similarly, by proceeding exactly as in Art. 18·5, it can be shown that the equation of the pair of tangents to the parabola $y^2 = 4ax$ from any point $P \equiv (x_1, y_1)$ outside it is $SS_1 = T^2$;[11] and so on.[12] We shall, therefore, discuss those topics

[10] See Ex. 1 of Art. 18 where the result has been obtained for the convenience of students (*i.e.*, to help them in getting a mastery over the necessary analysis).

[11] See Ex. 13 of Art. 18 where this equation has been obtained.

[12] The details have not, however, been worked out in all cases. This has been purposely done as it is hoped that *a careful perusal*

connected with tangents to a parabola which are of some special interest to us just now.[13]

30·1. To find the condition that the line $y = mx + c$ may touch the parabola $y^2 = 4ax$.

The equation of the tangent at any point (x_1, y_1) on the parabola is $yy_1 = 2a(x + x_1)$.

Now, if the given line touches the parabola at (x_1, y_1), then it must be the same as this line.

$$\therefore \qquad \frac{2a}{m} = \frac{y_1}{1} = \frac{2ax_1}{c}.$$

$$\therefore \qquad x_1 = \frac{c}{m} \text{ and } y_1 = \frac{2a}{m}.$$

And since (x_1, y_1) is on the parabola, we have $y_1^2 = 4ax_1$.

$$\therefore \qquad \frac{4a^2}{m^2} = \frac{4ac}{m},$$

$$\text{or} \qquad c = \frac{a}{m},$$

which is the required condition.[14-15]

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of the topics discussed and the examples worked out (for illustrative purposes) under Art. 18 will give the student that confidence in him which is essential for working out those results which have not been worked out for him.

[13] Similarly, we shall discuss those topics connected with *normals*, etc. which would appear to us to be of special interest in connection with the parabola; and so on for other conics.

[14] Note that the method used is the same as that of Art. 18·3 where a similar question connected with the circle has been discussed.

[15] Hence, $y = mx + \frac{a}{m}$ is a tangent to the parabola $y^2 = 4ax$ for all values of m.

30·11. Notes on the above.

(i) The tangent at the point $(at^2, 2at)$ on the parabola $y^2 = 4ax$ is

$$y = \frac{x}{t} + at.$$

$$= mx + \frac{a}{m}, \text{ where } m = \frac{1}{t}.$$

This shows that the point of contact of the tangent

$$y = mx + \frac{a}{m} \text{ is at } \left(\frac{a}{m^2}, \frac{2a}{m}\right).$$

(ii) Again, since $m = \frac{1}{t}$, and m = the tangent of the angle which the upward direction of the line $y = mx + \frac{a}{m}$ makes with the x-axis, t = the cotangent of the same angle.

30·2. On the number of tangents from a given point.

Let the equation of the parabola be $y^2 = 4ax$, and let $P \equiv (\alpha, \beta)$ be the given point.

The equation of any tangent to the parabola is

$$y = mx + \frac{a}{m}.$$

If it passes through the point P, we must have

$$\beta = m\alpha + \frac{a}{m}.$$

$$\therefore \quad m^2\alpha - m\beta + a = 0.$$

As this is a quadratic equation in m, it is clear that *two tangents (real or imaginary) can be drawn to a parabola from a given point outside it.*

These tangents will be *real, coincident*, or *imaginary* according as $\beta^2 - 4a\alpha >, =$, or <0.

But this is so according as P is *without, on*, or *within* the parabola. Hence, *the tangents will be real, coincident or imaginary according as the given point is without, on, or within the parabola.*[16]

30·3. Some geometrical theorems.

(*i*) *If the tangent at any point P on a parabola meets the axis at T, and PN is the ordinate at P, then TN is bisected at the vertex A.* [17]

Let the equation of the parabola be $y^2 = 4ax$ with reference to the axis and the tangent at the vertex as the axes of co-ordinates.

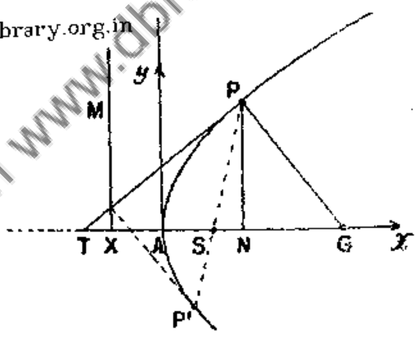

FIG 36

The tangent at $P \equiv (at^2, 2at)$ is $y = \frac{x}{t} + at$.

It meets $y = 0$ where $x = -at^2$.

$$\therefore \qquad AT = -at^2.$$

And since the abscissa of $P \equiv AN = at^2$, we have

$$TA = AN.$$

Hence, the *subtangent* TN is bisected at A.

---

[16] Note that the method used is exactly the same as that of Art. 18·4.

[17] The portion TN of the axis (of the parabola) as defined above is called the *subtangent* at the point P on the parabola (fig. 36).

(*ii*) *To show that* $TS = SP$ (fig. 36).

We have $SP = \sqrt{(at^2 - a)^2 + 4a^2t^2} = at^2 + a.$

And $TS = TA + AS = at^2 + a$, by (i).

Hence the result.[18]

Examples.

1. Write down the equations of the tangents at the ends of the latus-rectum of the parabola $y^2 = lx$.

2. Write down the equation of the tangent at any point 't' on the parabola $x = at^2$, $y = 2at$ (the axes being rectangular).

[*Hints.* As usual, write down the equation of the tangent at any point (x_1, y_1). Then, put $x_1 = at^3$, $y_1 = 2at$ in it and simplify, if necessary.][19]

**3. Find the equation of the tangent to the parabola $(ax + by)^2 = 2gx + c$ at the point (x_1, y_1) on it.

**4. Find the equation of the tangent to the curve $ax^2 + by^2 = 1$ at the point (x_1, y_1) on it.

5. Show that $y = mx + c$ touches the parabola $y^2 = lx$, if $l = 4mc$.

For any curve $y = f(x)$, the term *subtangent* means the portion TN of the x-axis (which is not necessarily an axis of the curve). For fuller informations on this point, see Edwards' *Differential Calculus*, Ch. VIII.

[18] As in the case of other geometrical theorems already proved, we have here followed the most straight-forward procedure of verifying the truth of the theorem with the help of analysis (see foot-note 15 of Ch. III).

[19] By Differential Calculus, the equation of the tangent is $y - 2at = m(x - at^2)$, where $m = \frac{dy}{dt} / \frac{dx}{dt}$.

6. Show that the line $x+y=1$ touches the parabola $y=x-x^2$.

7. *Show that any tangent to a parabola makes equal angles with the axis and the focal radius vector to the point of contact.*

8. *Show that the tangent at the vertex of a parabola is perpendicular to its axis.*

9. *The ordinate of the point of intersection of two tangents to a parabola is an arithmetic mean between the ordinates of the points of contact of the tangents.*

10. *The line through the focus of a parabola perpendicular to any tangent to it meets it on the tangent at the vertex.*

*11. Find the equation of the tangent at any point (x_1, y_1) on the curve $F(x, y) \equiv ax^2+2hxy+by^2+2gx+2fy+c=0$, the axes being rectangular or oblique.

[*Solution.* The equation of a line through $P \equiv (x_1, y_1)$

is $$\frac{x-x_1}{l}=\frac{y-y_1}{m}=r,$$

where r is the distance from P of any point Q on it and $l^2+m^2+2lm\cos\omega=1$ (ω being the angle between the axes).

The point Q is given by

$$x=x_1+lr \text{ and } y=y_1+mr.$$

If it lies on $F(x, y)=0$, we must have

$$F(x_1+lr, y_1+mr)=0.$$

$$\therefore \quad F(x_1, y_1)+r\left(l\frac{\delta F}{\delta x_1}+m\frac{\delta F}{\delta y_1}\right)+\frac{r^2}{2}$$

$$\times\left(l\frac{\delta}{\delta x_1}+m\frac{\delta}{\delta y_1}\right)^2 F=0. \text{ [20]}$$

[20] A knowledge of the *Differential Calculus* tells us that the result of substitution can be written in this form. In view of the limited

Now, if the line touches the given curve at $P \equiv (x_1, y_1)$, then both the roots of this equation in r must be zero. The conditions for this are

$$F(x_1, y_1) = 0 \text{ and } l\frac{\delta F}{\delta x_1} + m\frac{\delta F}{\delta y_1} = 0.$$

Eliminating $l : m$ from the last of these equations and the equation of the line, we have

$$(x - x_1)\frac{\delta F}{\delta x_1} + (y - y_1)\frac{\delta F}{\delta y_1} = 0$$

as the equation of the tangent.][21, 22]

31.

ON CHORDS.

Let the equation of the parabola be $y^2 = 4ax$.

Then, by using the methods of Art. 19, the following results can be easily obtained :—

(i) The equation of the chord joining the points (x_1, y_1) and (x_2, y_2) on the parabola is

$$(y - y_1)(y - y_2) = y^2 - 4ax.$$ [23]

scope of the book, it is, however, sufficient for us to note that the result of substitution can be written in the form $A + Br + Cr^2 = 0$.

[21] If we wish to put it in another form, we should proceed as in foot-note 14 of Ch. III. It will be found that $x\frac{\delta F}{\delta x_1} + y\frac{\delta F}{\delta y_1} + z\frac{\delta F}{\delta z_1} = 0$ (where $z = z_1 = 1$) is a convenient form of it.

[22] Note that the method used is the same as that of Art. 18·2 (i). Unlike the method of Art. 18·2 (ii), it can be used even when the axes are *oblique*.

[23] See Ex. 1 of Art. 19 where it has been worked out for the convenience of students.

(ii) The equation of the chord whose middle point is at $P\equiv(x_1, y_1)$ is given by

$$T = S_1,$$

where T and S_1 are as usual.[24]

(iii) The equation of the chord of contact of tangents from the point (x_1, y_1) is

$$yy_1 = 2a(x + x_1).\text{[25]}$$

31·1. A geometrical theorem.

Tangents at the extremities of a focal chord of a parabola intersect at right angles on the directrix.

Let the equation of the parabola be $y^2 = 4ax$; and let $P\equiv(x_1, y_1)$ be the point of intersection of the tangents at the extremities of a focal chord.

The chord of contact of tangents from $P\equiv(x_1, y_1)$ is

$$yy_1 = 2a(x + x_1).$$

As it passes through the focus $S\equiv(a, 0)$, we must have

$$2a(x_1 + a) = 0.$$

But $a \neq 0$. $\qquad \therefore \quad x_1 + a = 0. \qquad \ldots \qquad \ldots \qquad (1)$

Now, the equation of the pair of tangents from $P\equiv(x_1, y_1)$ is $SS_1 = T^2$,

or $(y^2 - 4ax)(y_1{}^2 - 4ax_1) = (yy_1 - 2ax - 2ax_1)^2$.

If these lines are perpendicular to each other, we must have

$$y_1{}^2 - 4ax_1 = y_1{}^2 + 4a^2.$$

$$\therefore \qquad 4a(x_1 + a) = 0.$$

But $a \neq 0$. $\qquad \therefore \quad x_1 + a = 0.$

---

[24] See Ex. 5 (ii) of Art. 19 (T and S_1 having the same significance as in foot-note 31 of Ch. III).

[25] See Ex. 6 of Art. 19 where it has been worked out.

But, by (1), this is true. Hence, it is clear that *the tangents at the extremities of a focal chord intersect at right angles.*

Again, the equation of the directrix is $x+a=0$. Hence, P will lie on the directrix, if $x_1+a=0$. But, by (1), this is *true*. Hence, *P lies on the directrix*. Hence the theorem.

Examples.

**1. Find the equation of the chord joining the points (x_1, y_1) and (x_2, y_2) on the parabola $y=x-x^2$.[26]

*2. Find the equation of a chord of $F(x, y) \equiv ax^2 + 2hxy + by^2 + 2fy + 2gx + c = 0$ in terms of the co-ordinates of its middle point, the axes being *rectangular or oblique*.

[*Solution*. The equation of a line through $P \equiv (x_1, y_1)$ is

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = r, \qquad \dots \quad (2)$$

where r is the distance from P of any point Q on it and $l^2 + m^2 + 2lm\cos\omega = 1$ (ω being the angle between the axes).

Now, Q is given by

$$x = x_1 + lr \text{ and } y = y_1 + mr.$$

If it lies on $F(x, y) = 0$, then we must have

$$F(x_1 + lr,\ y_1 + mr) = 0.$$

$$\therefore \quad F(x_1,\ y_1) + r\left(l\frac{dF}{dx_1} + m\frac{dF}{dy_1}\right) + \frac{r^2}{2}\left(l\frac{d}{dx_1} + m\frac{d}{dy_1}\right)^2 F = 0.$$

---

[26] Note that the equation of the chord joining the points (x_1, y_1) and (x_2, y_2) on $S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is $a(x-x_1)(x-x_2) + h(x-x_1)(y-y_2) + h(x-x_2)(y-y_1) + b(y-y_1)(y-y_2) = S$.

Hence, P will be the middle point of the chord whose equation is (2), if

$$l\frac{dF}{dx_1}+m\frac{dF}{dy_1}=0.$$

Eliminating $l:m$ from (2) and this equation, we have

$$(x-x_1)\frac{dF}{dx_1}+(y-y_1)\frac{dF}{dy_1}=0$$

as the equation of the required chord].[27-28]

3. If a chord of the parabola $y^2=lx$ subtends a right angle at the vertex, the tangents at the extremities of the chord meet on the line $x+l=0$.

4. Q is the middle point of a focal chord PSP' of a parabola, and QR is drawn perpendicular to PP' to cut the axis in R; show that

$$2SR=PS+SP'.$$

32.

ON NORMALS.

32·1. To find the equation of the normal to the parabola $y^2=4ax$ at the point $P\equiv(at^2, 2at)$ on it.

The equation of any straight line through $(at^2, 2at)$ is

$$y-2at=m(x-at^2)\;; \qquad \ldots \qquad \ldots \qquad (1)$$

and that of the tangent at P is

$$ty=x+at^2. \qquad \ldots \qquad \ldots \qquad (2)$$

---

[27] Writing it at full length, it can easily be seen to be same as $T=S_1$.

[28] Note that the form of this is the same as that of the tangent at (x_1, y_1) on the curve $F(x, y)=0$ (see Ex. 11 of Art. 30). But their simplified forms cannot be the same, as (x_1, y_1) is not on the curve in both the cases.

If (1) and (2) are perpendicular to each other, we must have

$$m = -t.$$

Putting this value of m in (1), we have

$$y - 2at = at^3 - xt,$$

or $$y + xt = 2at + at^3 \quad \ldots \quad \ldots \quad (3)$$

as the equation of the normal.[29]

32·2. On the number of normals from a given point.

Let the equation of the parabola be $y^2 = 4ax$; and let $P \equiv (x_1, y_1)$ be the given point.

By Art. 32·1, the equation of the normal at 't' is

$$y + xt = 2at + at^3.$$

It will pass through the given point $P \equiv (x_1, y_1)$, if

$$y_1 + tx_1 = 2at + at^3.$$

As this is an equation of the third degree in t, it is clear that three normals (two of which may be imaginary) can be drawn to a parabola from a given point outside it.

32·21. Some geometrical theorems.

(*i*) *If the normals at any three points on a parabola meet in a point, then the sum of the ordinates of the three points is zero.*[30]

Let the equation of the parabola be $y^2 = 4ax$.

The normal at any point 't' is

$$y + xt = 2at + at^3.$$

It passes through a given point $P \equiv (x_1, y_1)$, if

$$y_1 + x_1 t = 2at + at^3. \quad \ldots \quad \ldots \quad (4)$$

[29] Note that the method used is the same as that of Art. 20·2.

[30] The term *ordinate* of a parabola has been defined in Art. 29·3 (i).

Let the roots of this equation in t be t_1, t_2 and t_3. Then, the normals at the points 't_1', 't_2', 't_3' pass through given point P.

Now, from (4), we have

$$t_1 + t_2 + t_3 = 0.$$

$$\therefore \quad 2a(t_1 + t_2 + t_3) = 0.$$

$$\therefore \quad y_1 + y_2 + y_3 = 0,$$

where y_1, y_2, y_3 are the ordinates of these points. Hence the theorem.

(*ii*) *If the normal at any point on a parabola meets its axis in G, and PN is the ordinate at P, then NG = a constant.* [31]

Let the equation of the parabola be $y^2 = 4ax$.

Now, the equation of the normal at any point 't' is

$$y + xt = 2at + at^3.$$

It meets the axis $y = 0$ where $x = 2a + at^2$.

$\therefore \quad AG = 2a + at^2$, (fig. 36).

Again, AN = the abscissa of $P = at^2$.

$\therefore \quad$ the subnormal $NG = AG - AN = 2a$ = a constant.

Hence the theorem.

Examples.

1. Write down the equation of the normal at (x_1, y_1) on the parabola $y^2 + kx = 0$.

**2. Find the equation of the normal at (x_1, y_1) on the parabola $(ax + by)^2 = 2gx + c$.

---

[31] The portion NG of the axis is called the *subnormal* at P on the parabola (fig. 36). For any curve $y = f(x)$, the term *subnormal* means the portion NG of the axis of x (which is not necessarily an axis of the curve). For further informations, see Edwards' *Differential Calculus*, Ch. VIII.

3. The normal at the point 't' meets the parabola $y^2 = 4ax$ again at the point 't_1'; show that

$$t_1 = -\left(t + \frac{2}{t}\right).$$

4. The normal chord at the point on the parabola $y^2 = 4ax$ where the co-ordinates are equal subtends a right angle at the focus.

5. Prove that the chord of the parabola $y^2 = lx$ which is normal at the point whose abscissa is $\frac{l}{2}$ subtends a right angle at the vertex.

**6. The normals at the points where the line $y = mx + c$ cuts the parabola $y^2 = 4ax$ meet at P. Find the co-ordinates of the third point on the parabola the normal at which will also pass through P.

[*Solution.* Let the line $y = mx + c$ cut the parabola at points 't_1' and 't_2'.

Then, it must be the same as the line joining the points 't_1' and 't_2'. Now, the equation of this line is

$$2x = (t_1 + t_2)y - 2at_1t_2.$$

$$\therefore \quad \frac{1}{t_1 + t_2} = \frac{m}{2} = \frac{c}{2at_1t_2}. \quad \ldots \quad \ldots \quad (5)$$

And if t_3 be the third point on the parabola the normal at which passes through $P \equiv (x_1, y_1)$, then t_1, t_2, t_3 must be the roots of the equation

$$y_1 + x_1t = 2at + at^3.$$

$$\therefore \quad t_1 + t_2 + t_3 = 0, \; t_1t_2t_3 = \frac{y_1}{a}, \text{ etc.}$$

$\therefore$ (5) becomes $\dfrac{1}{-t_3} = \dfrac{m}{2} = \dfrac{ct_3}{2y_1}$.

$$\therefore \qquad t_3 = -\frac{2}{m}.$$

Hence, the third point is $\left(\dfrac{4a}{m^2}, -\dfrac{4a}{m}\right)$.]

7. If the three normals that can be drawn to the perabola $y^2 = 4ax$ from a given point cut its axis at points whose distances from the vertex are in $A.P.$, then the point lies on the curve $27ay^2 = 2(x-2a)^3$.

[*Hints.* If the normals at 't_1', 't_2', 't_3' meet at $P \equiv (x_1, y_1)$, then t_1, t_2, t_3 must be the roots of the equation $y_1 + tx_1 = 2at + at^3$. Hence,

$$\Sigma t_1 = 0, \ \Sigma t_1 t_2 = \frac{2a - x_1}{a} \text{ and } t_1 t_2 t_3 = \frac{y_1}{a}.$$

The normals at t_1, t_2 and t_3 cut the axis at points whose distances from the vertex are $2a + at_1^2$, $2a + at_2^2$ and $2a + at_3^2$ respectively. Hence, by the condition of the problem, we have

$$2t_2^2 = t_1^2 + t_3^2.$$

Now, eliminate t_1, t_2, t_3 from these four relations.]

8. Find the condition that the normals at the extremities of a chord of the parabola $y^2 = 4ax$ may intersect at a point on the curve.

9. The normals at the points P, Q, R of a parabola meet in a point; show that the circle through P, Q and R passes through the vertex of the parabola.

33.

ON POLES AND POLARS.

Let the equation of the parabola be $y^2=4ax$.

Then, by using the methods of Art. 21, we can obtain the following results very easily :—

(i) The polar of a point $P\equiv(x_1, y_1)$ with respect to the parabola is $yy_1=2a(x+x_1)$.[32]

(ii) The pole of the line $y=mx+c$ with respect to the parabola is $\left(\frac{c}{m}, \frac{2a}{m}\right)$.[33]

(iii) If the polar of P passes through Q, then will the polar of Q pass through P.[34]

33·1. A geometrical theorem.

The polar of the focus with regard to a parabola is its directrix.

Let the equation of the parabola be $y^2=4ax$.

Then, its focus is $S\equiv(a, 0)$.

Now, the equation of the polar of S is $x+a=0$.

But this is also the equation of the directrix of the parabola. Hence the theorem.

Examples.

1. Write down the equation of the polar of (x_1, y_1) with respect to the parabola $y^2=l(x+a)$.

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[32] See Ex. 1 of Art. 21 where it has been worked out, and note its form.

[33] See Ex. 4 of Art. 21 where it has been worked out.

[34] See Ex. 7 of Art. 21 where it has been worked out.

**2. Show that the pole of the chord joining the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ on the parabola $y^2=4ax$ is $[at_1t_2, a(t_1+t_2)]$.

3. P is the pole of the chord QQ', and perpendiculars from Q, P and Q' are drawn on any tangent to a parabola; show that the lengths of these perpendiculars are in *G.P.*

[*Hints.* The tangent at any point T on the parabola $y^2=4ax$ is $ty=x+at^2$.

By Ex. 2, the pole of the chord joining $Q\equiv(at_1^2, 2at_1)$ and $Q'\equiv(at_2^2, 2at_2)$ is at $P\equiv\{at_1t_2, a(t_1+t_2)\}$.

Now, verify the truth of the theorem.]

4. Show that the condition that the lines $y=mx+c$ and $y=m'x+c'$ may be conjugate with respect to the parabola $y^2=4ax$ is $mc'+m'c=2a$.

**5. Find the polar of the point (x_1, y_1) with respect to the parabola $(\alpha x+\beta y)^2+2gx+2fy+c=0$.

34.

ON DIAMETERS.

34·1. On the locus of the middle points of a system of parallel chords of a parabola.

Let the equation of the parabola be $y^2=4ax$, and let the chords be ‖ to the line $y=mx$. Now, if $P\equiv(x_1, y_1)$ be the middle point of any chord of the parabola, its equation is $T=S_1$,

or $$yy_1-2ax=y_1^2-2ax_1.$$

If it is parallel to the given line, we must have

$$\frac{2a}{y_1}=m.$$

Hence, the locus of P is given by

$$y=\frac{2a}{m}.$$

[It is a straight line parallel to the axis of the parabola and is called a *diameter* of the parabola.][35]

34·11. Notes on the above.

Any chord bisected by a diameter of the parabola (or any conic) is called the *double ordinate* of that diameter.

Putting $y=\frac{2a}{m}$ in $y^2=4ax$, we have $x=\frac{a}{m^2}$.

Hence, a diameter of the parabola meets it in *only one* point at a finite distance from the origin. [36] The point where a diameter meets the parabola is called the *extremity* of that diameter.

34·2. A geometrical theorem.

(*i*) *The tangent at the extremity of a diameter is parallel to the chords bisected by that diameter.*

The equation of the diameter which bisects all chords parallel to the line $y=mx$ is $y=\frac{2a}{m}$.

---

[35] By a *diameter* of the parabola (or any conic) is meant the locus of the middle points of a system of parallel chords of the parabola (or the conic). [See Art. 23·1 (II).]

[36] That the other point of intersection is at an infinite distance from the origin can be easily seen as follows :—

The line $y=m_1x+c_1$ cuts the parabola at points whose abscissæ are given by $m_1{}^2x^2+(2m_1c-4a)x+c_1{}^2=0$.

Now, if the line be a *diameter*, we must have $m_1=0$ (since diameters are ∥ to the axis of the parabola). But, in this case, one of the roots of the above equation in x is *infinite*. Hence the result.

Its *extremity* is, therefore, at the point $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$.

The tangent at this point is given by

$$y = m\left(x + \frac{a}{m^2}\right);$$

and is evidently parallel to the line $y = mx$.

***34·3. To find the equation of the parabola, any diameter and the tangent at the extremity of that diameter being the axes of co-ordinates.**

Let the equation of the parabola be $y^2 = 4ax$ (the axis and the tangent at the vertex being the axes of co-ordinates); also let $P \equiv \left(\frac{a}{m^2}, \frac{2a}{m}\right)$ be any point on it.

Now, the tangent at P is given by

$$y = m\left(x + \frac{a}{m^2}\right).$$

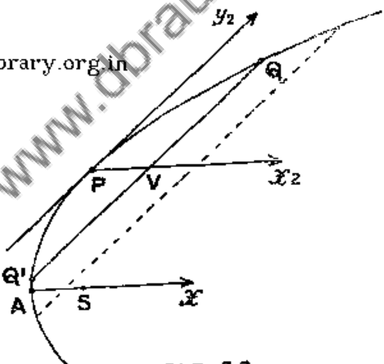

FIG. 37

Hence, it is inclined to the diameter (which is ∥ to the axis) at an angle such that $\tan \omega = m$.

Hence, to refer the parabola to the diameter and the tangent at P as axes of co-ordinates, it is sufficient to change the origin to P, and then turn the axis of y so that the new y-axis may be inclined at an angle ω to the axis of x (fig. 37).

Now, changing the origin to $P \equiv \left(\frac{a}{m^2}, \frac{2a}{m}\right)$, we have

$$\left(y_1 + \frac{2a}{m}\right)^2 = 4a\left(x_1 + \frac{a}{m^2}\right).$$

$$\therefore \quad y_1^2 + \frac{4ay_1}{m} = 4ax_1.$$

And turning the y-axis as stated above, we have (Art. 4·4)

$$y_2^2 \sin^2 \omega + 4ay_2 \cos \omega = 4a(x_2 + y_2 \cos \omega).$$

$$\therefore \qquad y_2^2 = \frac{4ax_2}{\sin^2 \omega}, \qquad \dots \quad (1)$$

or $y_2^2 = 4bx_2$, where $b = a/\sin^2 \omega$,
which is the required equation of the parabola.[37]

34·4. A geometrical theorem.

To show that $QV^2 = 4SP.\ PV$, *where* QV *is the ordinate of the diameter* PV (fig. 37).

Let $P \equiv \left(\frac{a}{m^2}, \frac{2a}{m}\right)$ be any point on the parabola $y^2 = 4ax$, the axes being rectangular.

Now, S is the point $(a, 0)$.

$$\therefore \quad SP^2 = \left(\frac{a}{m^2} - a\right)^2 + \frac{4a^2}{m^2} = a^2\left(1 + \frac{1}{m^2}\right)^2.$$

$$\therefore SP = a\left(1 + \frac{1}{m^2}\right) = a(\cot^2 \omega + 1) = a/\sin^2 \omega,$$

ω being the angle which the tangent at P makes with the axis of x.

Now, the equation of the parabola with reference to the diameter and the tangent at P is

$$y^2 = 4bx, \text{ where } b = a/\sin^2 \omega.$$

$$\therefore \qquad QV^2 = 4\frac{a}{\sin^2 \omega}.\ PV = 4SP.\ PV.$$

---

[37] It is of the form $y^2 = 4bx$, b having a meaning different from that of a (Art. 29·2). b is called the *parameter* of the diameter Px_2.

Examples

1. Write down the equation of the diameter of the parabola $y^2=4x$ which bisects all chords parallel to the line $4x+y=1$.

2. Find the equation of the diameter of the parabola $y^2=4ax$ which bisects the chord joining the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$.

**3. Find the equation of the diameter of the parabola $F\equiv(\alpha x+\beta y)^2+2gx+2fy+c=0$ which bisects all chords parallel to the line $y=mx$.

[*Solution.* The equation of the chord whose middle point is at (x_1, y_1) is

$$(x-x_1)\frac{\delta F}{\delta x_1}+(y-y_1)\frac{\delta F}{\delta y_1}=0.$$

If it is parallel to the given line, we must have

$$m=-\frac{\delta F}{\delta x_1}\Big/\frac{\delta F}{\delta y_1}.$$

Hence, by definition, the equation of the required diameter is

$$\frac{\delta F}{\delta x}+m\frac{\delta F}{\delta y}=0.]$$

4. Show that the tangents at the extremities of a chord intersect on the diameter which bisects the chord.

**5. The tangents at the ends of a chord of the parabola meet the diameter through any point P on it in T, T', and Q is the point of intersection of the chord and the diameter; show that $PQ^2=PT.PT'$.

35.

ON SOME LOCUS PROBLEMS CONNECTED WITH THE PARABOLA.

35·1. The elimination method.

(1) *To find the locus of the middle points of normal chords of the parabola $y^2 = 4ax$.*

If $P \equiv (x_1, y_1)$ be the middle point of a chord, its equation is $T = S_1$,

or $\quad yy_1 - 2ax = y_1^2 - 2ax_1$. ... ... (1)

Again, if (1) is normal at the point 't' on the parabola, then it must be the same as

$$y + tx = 2at + at^3, \quad ... \quad ... \quad (2)$$

which is the equation of the normal at 't'.

Comparing (1) and (2), we have

$$\frac{y_1}{1} = \frac{-2a}{t} = \frac{y_1^2 - 2ax_1}{2at + at^3}. \quad ... \quad ... \quad (3)$$

Now, eliminating t from the equations (3), we have

$$y_1^4 + 4a^2y_1^2 + 8a^4 = 2ax_1y_1^2.\ ^{*8}$$

This is a relation between the co-ordinates of any point P on the locus, and is *free* from any other unknown quantity.

Hence, the equation of the locus of P is

$$y^2(y^2 - 2ax + 4a^2) + 8a^4 = 0,$$

a curve of the *fourth* degree.

---

*8 To eliminate t, we have $t = -2a/y_1$, from the first two expressions of (3); and $y_1^2 - 2ax_1 = -2a^2(2 + t^2)$ from the last two. Now, writing $-2a/y_1$ for t in the latter equation, we have

$$y_1^2 - 2ax_1 = -2a^2\left(2 + \frac{4a^2}{y_1^2}\right). \quad \text{Hence the result.}$$

(II) *Find the locus of the poles of normal chords of the parabola $y^2 = 4ax$.*

The equation of the normal at any point 't' on the parabola is

$$y + tx = 2at + at^3. \quad \ldots \quad \ldots \quad (4)$$

If the pole of this line be $Q \equiv (x_1, y_1)$, then (4) must be the same as the polar of Q, *viz.*

$$yy_1 = 2a(x + x_1). \quad \ldots \quad \ldots \quad (5)$$

Comparing (4) and (5), we have

$$\frac{1}{y_1} = \frac{t}{-2a} = \frac{2t + t^3}{2x_1}. \quad \ldots \quad \ldots \quad (6)$$

Eliminating t from the equations (6), we find the condition which must be satisfied in order that Q may be a point on the locus.

To eliminate t, we have

$t = \dfrac{-2a}{y_1}$, from the first two expressions of (6);

and $x_1 = -a(2 + t^2)$, from the last two.

Now, substituting $-2a/y_1$ for t in the latter of these equations, we have

$$x_1 = -a\left(2 + \frac{4a^2}{y_1^2}\right).$$

Hence, the equation of the locus is

$$xy^2 + 2ay^2 + 4a^3 = 0,$$

a curve of the *third* degree. [39]

[39] In these problems, it is convenient (i) to obtain the equation of the line satisfying *only one* of the conditions, and then that of the line satisfying the *other*; and finally (ii) to obtain the conditions under which the two lines of (i) may be *identical*. This procedure reduces the problem to one of elimination. (Here the special feature elimination, not the use of parameters.)

35·2. The parametric method.

To find the locus of the middle points of chords of a parabola which subtend a right angle at its vertex.

Let the equation of the parabola be $y^2=4ax$, and let $P\equiv(at^2, 2at)$ and $Q\equiv(at_1^2, 2at_1)$ be the end-points of any chord PQ.

Now, the co-ordinates of the middle point of PQ are given by

$$\left.\begin{aligned} x &= \frac{a}{2}(t^2+t_1^2) \\ \text{and} \quad y &= a(t+t_1). \end{aligned}\right\} \quad \ldots \quad \ldots \quad (7)$$

Again, since the '*m*' of AP is $\frac{2}{t}$, and that of AQ is $\frac{2}{t_1}$ (A being the vertex), the condition that AP may be $\perp$ to AQ is $\quad tt_1=-4. \quad \ldots \quad \ldots \quad (8)$

Hence, the required locus is obtained by eliminating t and t_1, from (7) and (8).

To eliminate t and t_1, we have

$y^2=a^2(t^2+t_1^2+2tt_1)$, from the 2nd. equation of (7).

$\quad =2ax-8a^2$, by (7) and (8).

Hence, the locus is given by

$$y^2+8a^2=2ax.$$

[It is a parabola whose latus-rectum is half that of the original parabola.]

****35·3. The method of polar co-ordinates.**

To find the locus of the foot of the perpendicular from the vertex of a parabola to any tangent to it.

Let the equation of the parabola be $y^2 = 4ax$. Then, if

$$x \cos \alpha + y \sin \alpha - p = 0 \quad \ldots \quad (9)$$

be any tangent to it, we must have

$$a \sin^2 \alpha + p \cos \alpha = 0. [40] \quad \ldots \quad (10)$$

Now, the *polar* co-ordinates of the foot of the perpendicular from the vertex to the straight line (9) are p and α. Hence, since (10) is a relation between p and α, and is *free* from any other unknown quantity, the equation of the required locus is

$$a \sin^2 \theta + r \cos \theta = 0 . [41-42]$$

Examples.

**1. Find the locus of the point of intersection of tangents to a parabola which cut at a constant angle.

2. To find the locus of the middle points of chords of the parabola $y^2 = 4ax$ which pass through its focus.

3. Find the locus of the point of intersection of tangents to the parabola $y^2 = 4ax$ which intercept a constant length on the directrix.

---

[40] Comparing (9) with the equation of the tangent at (x_1, y_1), *viz.* $yy_1 = 2a(x + x_1)$, we have

$$\frac{\cos \alpha}{-2a} = \frac{\sin \alpha}{y_1} = \frac{p}{2ax_1} \quad \ldots \quad (11)$$

And since (x_1, y_1) is on the parabola, we have $y_1{}^2 = 4ax_1$.

Now, substituting in this equation the values of x_1 and y_1 as obtained from (11), we have $a \sin^2 \alpha + p \cos \alpha = 0$ which is the same as (10).

[41] This has been obtained from (10) by writing the current co-ordinates r and θ for p and α respectively.

[42] The Cartesian equation of the locus is $ay^2 + x(x^2 + y^2) = 0$. Hence, the locus is a curve of the *third* degree.

4. Find the locus of the middle points of chords of a parabola which are such that the normals at their extremities meet on the parabola.

5. *Tangents are drawn to the parabola $y^2 = 4ax$ from a given point P, and are inclined at angles θ_1, θ_2 to the axis of x; find the locus of P which moves so that $\tan^2\theta_1 + \tan^2\theta_2$ = a constant.*

[*Solution*. The tangent at any point 't' on the parabola is $yt = x + at^2$. If it passes through $P \equiv (\alpha, \beta)$, we must have

$$-\beta t + \alpha + at^2 = 0.$$

Now, let the roots of this equation in t be t_1 and t_2. Then, the tangents at 't_1' and 't_2' pass through d. Also

$t_1 + t_2 = \dfrac{\beta}{a}$ and $t_1\, t_2 = \dfrac{\alpha}{a}$.

But, by Art. 30·11, we have $t_1 = \cot\theta_1$ and $t_2 = \cot\theta_2$.

Hence, the condition of the problem becomes

$$\frac{1}{t_1^2} + \frac{1}{t_2^2} = k^2 \text{ (say)}.$$

Eliminating t_1 and t_2 from the three preceding equations, we find a relation between the co-ordinates of any point P on the locus and some known quantities.

To eliminate t_1 and t_2, we have $(t_1 + t_2)^2 - 2t_1t_2 = k^2 t_1^2 \times t_2^2$ from the last of the equations. Hence, $\beta^2 - 2a\alpha = k^2\alpha^2$ with the help of the other two equations. Hence, the equation of the locus is $y^2 = 2ax + k^2x^2$.]

6. Tangents are drawn to the parabola $y^2 = 4ax$ from a given point and are inclined at angles θ_1, θ_2 to the axis of x; find the locus of P when

(i) $\tan\theta_1 - \tan\theta_2$ = constant.

(ii) $\cot^2\theta_1 . \cot^2\theta_2$ = constant.

(iii) $\sin\theta_1 . \sin\theta_2$ = constant.

7. Find the locus of the point of intersection of normals at the extremities of chords passing through a given point.

** 8. Show that the locus of the poles of tangents to $y^2 = 2lx$ with respect to $x^2 + y^2 - lx = 0$ is the circle $2(x^2 + y^2) - lx = 0$.

** 9. Find the locus of a point which moves so that two of the normals from it to the parabola $y^2 = 4ax$ may cut at right angles.

[*Hints.* Let $P \equiv (x_1, y_1)$ be any point on the locus. If 't_1', 't_2', 't_3' be the points the normals at which pass through P, then t_1, t_2, t_3 are the roots of t in $y_1 + x_1 t = 2at + at^3$.

Hence, $\Sigma t_1 = 0$, $\Sigma t_1 t_2 = (2a - x_1)/a$ and $t_1 t_2 t_3 = y_1/a$. But, since two of the normals cut at right angles, we must have $t_1 t_2 = -1$ (say), by Art. 30·11.

Eliminating t_1, t_2, t_3 from these four equations, we have $y^2 = a(x - 3a)$ as the equation of the locus.]

10. Find the locus of the middle points of chords of the parabola $y^2 = 4ax$ which subtend a constant angle at the vertex.

** 11. Show that the locus of the poles of chords of the parabola $y^2 = 4ax$ which subtend a right angle at the focus given by is $x^2 - y^2 + 4ax + 4a^2 = 0$.

** 12. Find that the locus of the foot of the perpendicular from the focus to any normal to the parabola $y^2 = 4ax$.

36.

MISCELLANEOUS TOPICS CONNECTED WITH THE PARABOLA.

36·1. To find the length of the chord of contact of tangents from a given point P to the parabola $y^2=4ax$.

The equation of any tangent to the parabola is $yt=x+at^2$.

If it passes through the point $P\equiv(\alpha, \beta)$, we must have

$$\beta t=\alpha+at^2. \qquad \dots \qquad \dots \qquad (1)$$

If t_1 and t_2 be the roots of this equation in t, then the tangents at 't_1' and 't_2' must pass through P.

Also $t_1+t_2=\dfrac{\beta}{a}$ and $t_1 t_2=\dfrac{\alpha}{a}$. Now,

(the chord of contact)$^2=a^2(t_1^2-t_2^2)^2+4a^2(t_1-t_2)^2$.

$$=a^2(t_1-t_2)^2\{(t_1+t_2)^2+4\}.$$
$$=a^2\{(t_1+t_2)^2-4t_1t_2\}\{(t_1+t_2)^2+4\}.$$
$$=(\beta^2-4a\alpha)(\beta^2+4a^2)/a^2.$$

Hence the result.

***36·2. To find the equation of a parabola with reference to any two tangents as axes of co-ordinates.**

Consider the equation

$$\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}=1. \qquad \dots \qquad \dots \qquad (1)$$

Rationalizing, we have

$$\left(\frac{y}{b}-\frac{x}{a}+1\right)^2=\frac{4y}{b}. \qquad \dots \qquad \dots \qquad (2)$$

Now, the terms of the second degree in (2) form a perfect square. Hence, (1) represents a *parabola*.

Again, putting $x=0$, we have

$$\frac{y^2}{b^2}+\frac{2y}{b}+1=\frac{4y}{b}, \quad \text{or} \quad \left(\frac{y}{b}-1\right)^2=0.$$

Hence, the axis of y touches (*i.e.*, cuts at two coincident points) the parabola. Similarly, it can be shown that the axis of x touches the parabola. Hence, (1) is the equation of the parabola as required.

Examples.

1. Find the length of the chord of contact of tangents to the parabola $y^2=2x$ from the point (3, 4).

2. If the circle $x^2+y^2+ax+by+c=0$ cuts the parabola $y^2-lx=0$ in four points, the algebraic sum of the ordinates of these points will be zero.

** 3. Show that the area of the triangle formed by joining the feet of the normals from the point (α, β) to the parabola $y^2=4ax$ is

$$\{4a(\alpha-2a)^3-27a^2\beta^2\}^{\frac{1}{2}}$$

**4. R and S_1 are any two points on the parabola $y^2=4ax$, and $P\equiv(\alpha, \beta)$ is a fixed point on it. If RP and S_1P are perpendicular to each other, show that RS_1 passes through the fixed point $(\alpha+4a, -\beta)$.

5. Find the orthocentre of the triangle formed by joining three points on a parabola the normals at which pass through a given point.

**6. From any point on $y^2=a(x+m)$ tangents are drawn to $y^2=4ax$; show that the normals to the second parabola at the points of contact of these tangents intersect on a fixed straight line.

**7. Show that the circle circumscribing the triangle formed by three tangents to a parabola passes through its focus.

[*Hints.* Let tangents at points 't_1', 't_2' and 't_3' on the parabola $y^2=4ax$ form the triangle $T_1T_2T_3$. Then, the vertices of this triangle are $\{at_1t_2, a(t_1+t_2)\}$, $\{at_2t_3, a(t_2+t_3)\}$ and $\{at_3t_1, a(t_1+t_3)\}$.

Again, let the equation of the circle through T_1, T_2, and T_3 be $x^2+y^2+2gx+2fy+c=0$. Then, find the conditions under which T_1, T_2, T_3 may lie on the circle; and determine the values of g, f, and c from them. Finally, show that the focus $(a, 0)$ satisfies the equation of the circle thus found.]

*8. Find the condition that $lx+my=1$ may touch the parabola $\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}=1$.

**9. Find the equations of the common tangents to the parabolas $y^2=lx$ and $x^2=my$.

10. Show that two parabolas which have the same focus and their axes in opposite directions are at right angles to each other.

11. Find the angle at which the parabolas $y^2=lx$ and $x^2=my$ cut each other.

CHAPTER VI.

THE ELLIPSE.

37.

ON THE ELLIPSE AND ITS EQUATION.

37·1. Definition and general equation of the ellipse.

For an ellipse, $e < 1$. Hence, an ellipse is the locus of a point P which moves so that its distance from a fixed point S bears a constant ratio (which is less than unity) to its distance from a fixed straight line.[1]

Let $P \equiv (x, y)$ and $S \equiv (x_1, y_1)$; also let the equation of the directrix be

$$x \cos \alpha + y \sin \alpha - p = 0.$$

Then, by definition, the equation of the ellipse is

$$(x - x_1)^2 + (y - y_1)^2 = e^2(x \cos \alpha + y \sin \alpha - p)^2, \quad \ldots \quad (1)$$

where $e < 1$.

37·2. To find the standard form of the equation of the ellipse.

Let S be the focus and MX be the directrix; and let $XS = d$ (fig. 38).

Also let XS and XM be taken as the axes of x and y respectively. Then, the focus, S is the point $(d, 0)$; and if $P \equiv (x, y)$ be any point on the ellipse, its equation is

$$(x - d)^2 + y^2 = e^2 x^2, \text{ where } e < 1.$$

---

[1] The fixed point is called the *focus*, and the fixed straight line is called the *directrix* of the ellipse (see Art. 25).

$$\therefore \qquad x^2(1-e^2)-2dx+y^2+d^2=0.$$

$$\therefore \qquad \left(x-\frac{d}{1-e^2}\right)^2+\frac{y^2}{1-e^2}=\frac{d^2e^2}{(1-e^2)^2}.$$

Now, changing the origin to the point $C \equiv \left(\frac{d}{1-e^2},\ 0\right)$ without changing the directions of the axes, we have

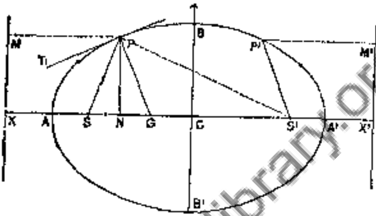

FIG. 38

$$x^2+\frac{y^2}{1-e^2}=\frac{d^2e^2}{(1-e^2)^2}. \qquad \dots \qquad \dots \qquad (2)$$

And writing a for $\frac{de}{1-e^2}$, and b^2 for $a^2(1-e^2)$ in (2), we have

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \qquad \dots \qquad \dots \qquad (3)$$

where a and b are *real* (e being < 1).

This is the equation of an ellipse in what is known as its *standard form*.

37·21. On the form of the curve whose equation is given by (3), the axes being rectangular.

Writing $a^2(1-e^2)$ for b^2, and $\frac{de}{1-e^2}$ for a (where $e < 1$) in (3), we have

$$x^2+\frac{y^2}{1-e^2}=\frac{d^2e^2}{(1-e^2)^2}.$$

Now, changing the origin to the point $\left(-\frac{d}{1-e^2},\ 0\right)$ we have

$$\left(x-\frac{d}{1-e^2}\right)^2+\frac{y^2}{1-e^2}=\frac{d^2e^2}{(1-e^2)^2}.$$

$$\therefore \qquad (x-d)^2+y^2=e^2x^2. \qquad \dots \qquad \dots \qquad (4)$$

Hence, the distance of the point $P \equiv (x, y)$ from the fixed point $S \equiv (d, 0)$ is in a constant ratio (which is less than unity) to its distance from the axis of y. Hence, the locus of P is an ellipse.

37·3. On some points connected with the ellipse.

(i) Since (3) remains unaltered if $-x$ is written for x, the curve is symmetrical about the axis of x. Hence, $y=0$ is an axis of the ellipse. Similarly, it can be seen that $x=0$ is also an axis of the ellipse.

Again, since the equation is unaltered by writing $-x$ for x and $-y$ simultaneously, the ellipse is symmetrical about the origin C.[2] Hence, every chord through C is bisected at it. This point C is called the *centre* of the ellipse.[3]

(ii) Putting $y=0$ in $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we have $x=\pm a$.

Hence, the points A and A' where the axis of x cuts the curve are distant a from C, and $AA'=2a$ (fig. 28).

Again, putting $x=0$ in the same equation, we have $y=\pm b$. Hence, the points B and B' where the axis of y cuts the curve are distant b from C, and $BB'=2b$.

Since, $b^2=a^2(1-e^2)$, we have $a>b$ (*i. e.*, the chord $AA'>$ the chord BB'). AA' is called the major axis of the ellipse, and BB' is called its *minor axis*.

If N be the foot of the perpendicular from any point P on the ellipse to its major axis, then PN is called the

[2] See foot-note 64 of Ch. I.

[3] By the *centre* of an ellipse (or any conic) is meant the point which is such that all chords passing through it are bisected at that point.

ordinate of the point P. The *double-ordinate* through the focus is called the *latus-rectum* (see **Art.** 29·3).

(iii) From (3), we have $y = \pm b\sqrt{1 - \dfrac{x^2}{a^2}}$.

Hence, in order that y may be real, x must lie between $-a$ and $+a$. Similarly, it can be seen that y must lie between $-b$ and $+b$ in order that x may be *real*. Hence, the curve is a *closed* one. (See foot-note 64 of Ch. I.)

(iv) The symmetry of the curve shows that there is a *second focus* S' on the major axis at the same distance from C as S; and that there is a second *directrix* (associated with S') parallel to the other directrix, and at the same distance from C as XM.

(v) From Art. 37 2, it is clear that

$$XC = \frac{d}{1-e^2} = \frac{a}{e} = \frac{AC}{e}.$$

$$\therefore \qquad CA = e.\,CX. \qquad \ldots \qquad \ldots \qquad (5)$$

$$\text{Again,} \qquad SC = XC - XS = \frac{de}{1-e^2} - d = \frac{de^2}{1-e^2} = ae.$$

$$\therefore \qquad CS = e.CA. \qquad \ldots \qquad \ldots \qquad (6)$$

Moreover, $SP + S'P = e.PM + e.PM'$ (fig. 38).

$= e.XX' = 2e.CX = 2CA$, by (5).

$= 2a$.

Hence, *the sum of the focal distances of any point on the ellipse is constant and equal to its major axis.* [4]

From (6), we see that the *focus* S is at the point $(-ae, 0)$ [or $-(\sqrt{a^2 - b^2}, 0)$]. The equation of the *directrix*

[4] Thus, we have a method of constructing an ellipse *mechanically* by means of a piece of thread (whose end points are fastened to two points fixed on a piece of paper) and a pencil.

is $x = -\frac{a}{e}$. The new focus S' is the point $(ae, 0)$ [or $(\sqrt{a^2 - b^2}, 0)$]. The new *directrix* is given by $x = \frac{a}{e}$.

Putting $x = -ae$ in (3), we have $y = \pm b\sqrt{1 - e^2} = \pm\frac{b^2}{a}$.

Hence, the *latus-rectum* (the double-ordinate through the focus) $= \frac{2b^2}{a}$.

**(vi) We have seen that the ellipse (3) has two foci $(\pm\sqrt{a^2 - b^2}, 0)$ on the *major axis*, and two directrices $x = \pm\frac{a^2}{\sqrt{a^2 - b^2}}$ corresponding to them. Now, from the symmetry of the equation (3), it is clear that there are still two other *foci* $(0, \pm\sqrt{b^2 - a^2})$ on the *minor axis*, and two *directrices* $y = \pm\frac{b^2}{\sqrt{b^2 - a^2}}$ corresponding to them.

But, since $b < a$ for the ellipse, it is clear that these foci and directrices are all *imaginary*. In this case, the *eccentricity* $= \frac{\sqrt{b^2 - a^2}}{b}$, and is also *imaginary*.

(vii) Putting $a = b$ in (3), we have $e = 0$. In this case, the equation becomes $x^2 + y^2 = a^2$, which is the equation of a circle of radius a (including a *point*, or a *pair of conjugate imaginary lines*, if $a = 0$). Hence, the circle (including the point, or a pair of conjugate imaginary lines) may be regarded as a special case ($e = 0$) of the ellipse.

**(viii) Changing the origin to the vertex $A \equiv (-a, 0)$ without changing the directions of the axes, we have

$$\frac{(x-a)^2}{a^2} + \frac{y^2}{b^2} = 1.$$

$$\therefore \quad \frac{x^2}{a} - 2x + \frac{y^2}{a(1-e^2)} = 0. \qquad \dots \qquad \dots (7)$$

Putting $a = \frac{k}{1-e}$ in (7), we have

$$\frac{x^2(1-e)}{k} - 2x + \frac{y^2}{k(1+e)} = 0.$$

Now, supposing that k remains *finite* while e tends to unity, we see that the ellipse approximates to $y^2 = 4kx$, which is a parabola. Hence, it is clear that the parabola may be regarded as a *limiting case* of the ellipse.[5]

37·4. To find the condition that an equation of the second degree in x, y may represent an ellipse.

The most general equation of the second degree in x, y is

$$ax^2 + 2hxy + by^2 + 2fy + 2gx + c = 0, \qquad \dots \qquad \dots (8)$$

where a, h, b, f, g and c are arbitrary constants (none being *zero*).

Comparing (8) with (1), we have

$$\frac{a}{1-e^2\cos^2\alpha} = \frac{h}{-e^2 \sin\alpha \cos\alpha} = \frac{b}{1-e^2\sin^2\alpha} = \text{etc.}$$
$$= k \text{ (say)}. \qquad \dots \qquad (9)$$

[5] Besides giving other informations, the results of this article enable us to find the equations of the *axes* and the *directrix*, the *eccentricity*, the co-ordinates of the *focus*, the lengths of the *axes*, and the *latus-rectum* of the ellipse (as represented by an equation of the second degree in x, y) by reducing its equation to the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (the axes being rectangular). [See foot-note 4 of Ch. V.]

From the first two equations of this set, we have

$$\frac{ab-h^2}{h^2}=(1-e^2\cos^2\alpha)(1-e^2\sin^2\alpha)-e^2\sin^2\alpha\cos^2\alpha.$$

$$=1-e^2=\text{a positive quantity, since } e<1$$

for an ellipse.

Hence, we have $ab-h^2>0$ as a necessary condition.

37·41. On the sufficiency of the condition $ab-h^2>0$.

The equation of any conic may be written as

$$(x-x_1)^2+(y-y_1)^2=e^2(x\cos\alpha+y\sin\alpha-p)^2,$$

(where e is not necessarily <1).

Comparing this with (8), we have

$$\frac{a}{1-e^2\cos^2\alpha}=\frac{h}{-e^2\sin\alpha\cos\alpha}=\frac{b}{1-e^2\sin^2\alpha}=\text{etc}$$

But, $ab-h^2>0$ for an ellipse.

$\therefore\quad (1-e^2\cos^2\alpha)(1-e^2\sin^2\alpha)-e^4\sin^2\alpha\cos^2\alpha>0.$

$\therefore\quad e^2<1$, which shows that the conic is an *ellipse*.

Hence, the condition $ab-h^2>0$ is *sufficient* to make (8) represent an ellipse.

37·42. Note on the conditions necessary to determine an ellipse.

From the above, it is clear that five conditions are *necessary* and *sufficient* to determine an ellipse, these conditions being such that each may give rise to one equation between the constants. Thus, an ellipse may be made to pass through any five points (no three of which are in one straight line).

****37·5. To find the centre of the conic**

$$F(x, y) \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0. \quad \ldots \quad (10)$$

The equation of a line through $C \equiv (x_1, y_1)$ making an angle θ with the axis of x is

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r,$$

where r is the algebraic distance of any point $Q \equiv (x, y)$ on it from C.

The point Q is given by

$$x = x_1 + r\cos\theta \text{ and } y = y_1 + r\sin\theta.$$

If this point lies on (1), we must have

$$F(x_1 + r\cos\theta, y_1 + r\sin\theta) = 0.$$

$$\therefore \quad F(x_1, y_1) + r\left(\cos\theta \frac{\delta F}{\delta x_1} + \sin\theta \frac{\delta F}{\delta y_1}\right) + \frac{r^2}{2}$$

$$\times \left(\cos\theta \frac{\delta}{\delta y_1} + \sin\theta \frac{\delta}{\delta y_1}\right)^2 F = 0.$$ [6]

Now, if $C \equiv (x_1, y_1)$ be the centre of the conic, then every straight line through it must meet the curve in two points equidistant from C. Hence, the roots of this equation in r must be equal and opposite for all values of θ. The condition for this is

$$\cos\theta \frac{\delta F}{\delta x_1} + \sin\theta \frac{\delta F}{\delta y_1} = 0$$

for all values of θ. But this is impossible unless

$$\left.\begin{aligned} \frac{\delta F}{\delta x_1} &= 0, \\ \text{and } \frac{\delta F}{\delta y_1} &= 0. \end{aligned}\right\} \quad \ldots \quad \ldots \quad (11)$$

[6] See foot-note 20 of Ch. V.

Hence, the co-ordinates of the centre are given by (11).

Now, writing these equations in full, we have

$$ax_1 + hy_1 + g = 0,$$
$$hx_1 + by_1 + f = 0.$$

And solving them for x_1 and y_1, we have

$$\left.\begin{aligned} x_1 &= \frac{hf - bg}{ab - h^2}, \\ y_1 &= \frac{gh - af}{ab - h^2}. \end{aligned}\right\} \quad \ldots \quad \ldots \quad (12)$$

**37·51. Notes on the above.

The equations (12) are the same as those which give the point of intersection of the straight lines when (10) represents a pair of such lines. When $ab - h^2 = 0$ (*i.e.*, when the curve is a parabola), both x_1 and y_1 become infinitely great, except when $\frac{a}{h} = \frac{h}{b} = \frac{g}{f}$ in which case the curve degenerates into a pair of coincident straight lines. But x_1, y_1 are *finite* when $ab - h^2 \neq 0$ in which case the conic is either an ellipse, or a hyperbola. For this reason, an ellipse and a hyperbola are called *central conics*; while a parabola (whose centre is at infinity) is called a *non-central conic.*

**37·6. On the central conic as represented by the general equation of the second degree.

The most general equation of the second degree is

$$F(x, y) \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, \quad \ldots \quad (13)$$

where a, h, b, g, f and c are arbitrary conatants (none being *zero*).

[It represents a central conic, if $ab \neq h^2$.]

By Art. 37·5, the co-ordinates (x_1, y_1) of the centre of the conic are given by

$$\left.\begin{aligned} ax_1 + hy_1 + g = 0, \\ hx_1 + by_1 + f = 0. \end{aligned}\right\} \quad \ldots \quad \ldots \quad (14)$$

$$\left.\begin{aligned} \therefore \quad x_1 &= \frac{hf - bg}{ab - h^2} \\ \text{and } y_1 &= \frac{gh - af}{ab - h^2}. \end{aligned}\right\} \quad \ldots \quad \ldots \quad (15)$$

Now, transferring the origin to (x_1, y_1) without changing the directions of the axes, we have

$$x = X + x_1 \text{ and } y = Y + y_1.$$

Hence, (13) becomes $F(X + x_1, Y + y_1) = 0$.

$$\therefore \quad aX^2 + 2hXY + bY^2 + c' = 0,$$

where $c' \equiv ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c$.

$$= x_1(ax_1 + hy_1 + g) + y(hx_1 + by_1 + f) + gx_1 + fy_1 + c.$$

$$= gx_1 + fy_1 + c, \text{ by (14).}$$

$$= \frac{\Delta}{C}$$

$$\text{where } \Delta \equiv \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix} \text{ and } C \equiv ab - h^2.$$

Hence, the equation of the conic with its centre as the origin is

$$aX^2 + 2hXY + bY^2 + \frac{\Delta}{C} = 0.$$

Hence, it is *sufficient* to discuss an equation of the form $ax^2 + 2hxy + hy^2 = 1$ for further informations on the subject.

****37·61. To find the lengths and position of the axes of the ellipse whose equation is $ax^2 + 2hxy + by^2 = 1$, $(ab > h^2)$.**

(i) If an ellipse be cut by any concentric circle, then the lines joining the origin to the points of intersection of the ellipse and the circle are *equally inclined to the axes of the ellipse.*[7] These lines will, therefore, be *coincident* when (and only when) the radius of the circle is equal to either of the semi-axes of the ellipse. Now, the equation of the pair of straight lines joining the origin to the points of intersection of the ellipse and the circle $x^2 + y^2 = r^2$ is

$$\left(a - \frac{1}{r^2}\right) x^2 + 2hxy + \left(b - \frac{1}{r^2}\right) y^2 = 0. \quad \ldots \quad (16)$$

These lines will be *coincident*, if

$$\left(a - \frac{1}{r^2}\right)\left(b - \frac{1}{r^2}\right) - h^2 = 0. \quad \ldots \quad (17)$$

Hence, the squares of the lengths of the semi-axes of the ellipse are given by the roots of r^2 in (17).[8-9]

---

[7] The truth of this result can be easily verified by taking $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ as the equation of the ellipse and $x^2 + y^2 = r^2$ as that of the concentric circle. For, the lines joining the origin to the points of intersection of the two curves are given by $x^2\left\{\frac{1}{a^2} - \frac{1}{r^2}\right\} + y^2\left\{\frac{1}{b^2} - \frac{1}{r^2}\right\} = 0$. And the bisectors of the angles between these lines are given by $xy = 0$ which is the equation of the axes of the ellipse. Hence, the lines joining the origin to the points of intersection of the ellipse and the circle are *equally* inclined to the axes. [Note that the result would be true even if the sign of a^2 or b^2 were changed.]

[8] Since the conic is an ellipse (real), both the roots of (17) in r^2 are *positive.*

[9] The lengths of the axes may also be obtained as follows :—

(ii) To find the position of the axes, we proceed as follows :—

Multiplying (16) by $a-\frac{1}{r^2}$, we have

$$\left(a-\frac{1}{r^2}\right)^2 x^2+2h\left(a-\frac{1}{r^2}\right)xy+h^2y^2=0,$$

if r^2 be a root of (17).

$$\therefore \qquad \left(a-\frac{1}{r^2}\right)x+hy=0.$$

Hence, if α^2 and β^2 be the roots of (17) in r^2, the equations of the corresponding axes are

$$\left(a-\frac{1}{\alpha^2}\right)x+hy=0 \text{ and } \left(a-\frac{1}{\beta^2}\right)x+hy=0,$$

respectively.[10]

Let the equation of the ellipse be reduced to its standard form $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=1$ by a rotation of axes. Then, by Ex. 10 of Art. 4, we have

$$\left.\begin{aligned}\frac{1}{\alpha^2}+\frac{1}{\beta^2}&=a+b,\\ \text{and}\quad \frac{1}{\alpha^2\beta^2}&=ab-h^2.\end{aligned}\right\} \quad \dots \quad \dots (18)$$

Hence, α^2 and β^2 (which are the squares of the semi-axes) are given by (18).

[10] The equation of the axes may also be obtained by turning the axes through an $\angle\theta$ so that the new equation may not contain any term in xy. But, in this case, it will not be possible for us to indicate the directions of the *major* and *minor* axes definitely.

Now, by Ex. 4 of Art. 4 and Art. 37·3, we know that if the axes are turned through an angle θ given by $\tan 2\theta=2h/(a-b)$, then the equation $ax^2+2hxy+by^2=1$ is reduced to $\frac{x_1^2}{\alpha^2}+\frac{y_1^2}{\beta^2}=1$. Hence, the directions of the axes are given by $\frac{\tan\theta}{1-\tan^2\theta}=\frac{h}{a-b}$. But, $\tan\theta=\frac{y}{x}$, where (x, y) is any point on either axis of the ellipse. Hence, the equation of the axes is $\frac{x^2-y^2}{a-b}=\frac{xy}{h}$.

****37·62. To find the eccentricity of the ellipse whose equation is $ax^2 + 2hxy + by^2 = 1$, $(ab > h^2)$.**

Let the axes be so turned that the equation may be reduced to its standard form

$$\frac{x_1^2}{\alpha^2} + \frac{y_1^2}{\beta^2} = 1$$

with its x_1-axis along the major axis (so that $\alpha^2 > \beta^2$).[11]

Then, by Ex 10 of Art. 4, we have

$$\left.\begin{aligned} \frac{1}{\alpha^2} + \frac{1}{\beta^2} &= a + b, \\ \text{and} \quad \frac{1}{\alpha^2\beta^2} &= ab - h^2. \end{aligned}\right\} \qquad \ldots \qquad \ldots \qquad \ldots (19)$$

Also $$e^2 = \frac{\alpha^2 - \beta^2}{\alpha^2} \qquad \ldots \qquad \ldots \qquad \ldots (20)$$

for the ellipse (when e is *real*).

Eliminating α^2 and β^2 from (19) and (20), we obtain a relation from which e^2 can be found in terms of a, h and b. To eliminate α^2 and β^2, we proceed as follows :—

From (19), we have

$$\alpha^2\beta^2 = \frac{1}{ab - h^2} \quad \text{and} \quad \alpha^2 + \beta^2 = \frac{a + b}{ab - h^2};$$

and from (2), we have $2 - e^2 = \dfrac{\alpha^2 + \beta^2}{\alpha^2}$.

$$\therefore \quad \frac{e^2}{2 - e^2} = \frac{\alpha^2 - \beta^2}{\alpha^2 + \beta^2} = + \frac{\sqrt{(\alpha^2 + \beta^2)^2 - 4\alpha^2\beta^2}}{\alpha^2 + \beta^2}$$

$$\therefore \quad \frac{e^2}{2 - e^2} = + \frac{\sqrt{(a - b)^2 + 4h^2}}{a + b}, \text{ by (19).}$$

Hence, e^2 can be obtained at once from the above relation.

---

[11] As it does not matter which axis is taken as the axis of x, we shall suppose that we have chosen the x-axis so that it may correspond to the root α^2.

****37·63. To find the foci of the ellipse whose equation is $ax^2+2hxy+by^2=1$, $(ab>h^2)$.**

If α^2 be the square of the length of the semi-major axis, its equation is

$$\left(a-\frac{1}{\alpha^2}\right)x+hy=0.$$

$$\therefore \quad y=\frac{1-a\alpha^2}{h\alpha^2}x=\tan\theta_1.x, \qquad \text{where } \tan\theta_1=\frac{1-a\alpha^2}{h\alpha^2}.$$

Now, since $CS=\sqrt{\alpha^2-\beta^2}$, the foci (on the major axis) are at the points $(\sqrt{\alpha^2-\beta^2}\cos\theta_1,\ \sqrt{\alpha^2-\beta^2}\sin\theta_1)$

and $(-\sqrt{\alpha^2-\beta^2}\cos\theta_1,\ -\sqrt{\alpha^2-\beta^2}\sin\theta_1)$.

[For an equation with numerical co-efficients, these quantities can be calculated quite easily. The two other foci (on the minor axis) are *imaginary*.]

***37·7. The position in oblique co-ordinates.**

By Ex. 12 of Art. 4, it is clear that the condition $ab-h^2>0$ is *necessary and sufficient* to make (13) represent an ellipse even when the axes are *oblique*.

37·8. Parametric equations of the ellipse.

As in Art. 29·7, it can be easily seen that the equations

$$\left.\begin{aligned} x&=a\cos\phi,\\ \text{and}\quad y&=b\sin\phi \end{aligned}\right\} \qquad \dots \qquad \dots (21)$$

form a set of *parametric* equations for the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. [See foot-note 8 of Ch. V]. ϕ is called the *eccentric angle* of the point (x, y) of the ellipse.

37·81. Geometrical interpretation of the eccentric angle.

Let APA' be an ellipse, and let AA' be its major axis (fig. 39).

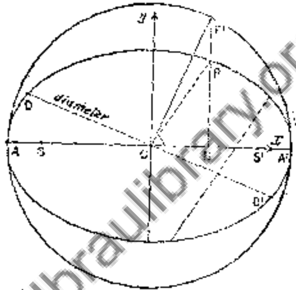

FIG. 39

Describe a circle on AA' as a diameter;[12] and let the ordinate NP of the point P on the ellipse be produced to meet the circle in P'. Then, since P is on the ellipse, we have

$$\frac{NP^2}{b^2} = 1 - \frac{CN^2}{a^2} = \frac{a^2 - CN^2}{a^2} = \frac{NP'^2}{a^2}.$$

$$\therefore \qquad NP' = NP.\ \frac{a}{b} = a \sin\phi, \text{ since } NP = b\sin\phi.$$

Hence, from the right-angled triangle CNP', it is clear that ϕ may be taken to be equal to the $\angle NCP'$ as defined above.[13]

Examples.

1. Write down the equation of the ellipse whose focus is at the point (2, 3), directrix is the line $3x - 2y + 1 = 0$, and eccentricity $= \frac{1}{2}$.

2. Write down the equation of the ellipse whose semi-axes are of lengths 3 and 2 respectively.

---

[12] This circle is called the *auxiliary circle* of the ellipse APA'. The points P and P' are said to form a pair of *corresponding points*.

[13] In what follows, we shall suppose that the *eccentric angle* may have any *positive* value (Cf. Art. 2·2 of Ch. I).

3. Find the equation of the ellipse whose latus-rectum = 6, and eccentricity = $\frac{2}{3}$.

4. Show that the point $Q \equiv (x_1, y_1)$ will lie *within*, *on*, or *without* the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ according as $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 <, =, \text{ or } > 0$

[*Hints.* Proceed as in Ex. 12 of **Art.** 29.]

5. Show that the equation $\frac{l}{r} = 1 - e \cos\theta$ represents an ellipse, if $e < 1$.

6. If a circle cuts the ellipse of Ex. 4 in four points, the sum of their eccentric angles is an even multiple of π.

7. In fig. 39, show that $\frac{PN^2}{AN.NA'} = \frac{b^2}{a^2}$ for the ellipse.

**8. Find the centre, and the lengths and position of the axes of the ellipse $3x^2 + 4xy + 3y^2 + 4x - 4y = 2$.

[*Solution.* The *centre C* is given by $3x_1 + 2y_1 + 2 = 0$ and $2x_1 + 3y_1 - 2 = 0$. Hence, it is at the point $(-2, 2)$. Hence, referred to parallel axes through the centre, the equation of the conic is

$$3x^2 + 4xy + 3y^2 = 10.$$

By Art. 37·61 (*i*), the lengths of the semi-axes are given by

$$\left(\frac{3}{10} - \frac{1}{r^2}\right)^2 = \frac{1}{5^2}, \text{ since } a = \frac{3}{10} = b \text{ and } h = \frac{1}{5},$$

$\therefore \quad r^4 - 12r^2 + 20 = 0.$

$\therefore \quad r^2 = 2, \text{ or } 10.$

Again, by Art. 37·61 (*ii*), the equation of either axis (referred to ∥ axes through C) is

$$\left(\frac{3}{10} - \frac{1}{r^2}\right) x + \frac{1}{5} y = 0.$$

Hence, putting $r^2 = 10$ in the above equation, we have $x + y = 0$ as the equation of the *major axis*. Similarly, it can be shown that $y - x = 0$ is the equation of the *minor* axis.]

**9. Find the centre, and the lengths and position of the axes of an ellipse in the following cases :—

(i) $3x^2 + 2xy + 3y^2 = 5.$

(ii) $12x^2 + 8xy + 3y^2 - 18x - 16y = 8.$

(iii) $3x^2 + 4xy + 2y^2 + 4x + 12 = 0.$

**10. Find the latus-rectum, eccentricity and foci of the ellipse of Ex. 8.

[*Solution.* The *latus-rectum* $= \dfrac{2\beta^2}{\alpha} = \dfrac{4}{\sqrt{10}}.$ By Art. 37·62, the *eccentricity* e is given by

$$\frac{e^2}{2 - e^2} = + \frac{\sqrt{(a-b)^2 + 4h^2}}{a+b} = \frac{2}{3}.$$

$$\therefore \qquad e = \frac{2}{\sqrt{5}}.$$

To find the foci, we proceed as follows (Art. 37·63) :—

The equation of the major axis is $x + y = 0$.

$$\therefore \qquad \tan\theta_1 = -1.$$

$$\therefore \qquad \cos\theta_1 = -\frac{1}{\sqrt{2}} \text{ and } \sin\theta_1 = \frac{1}{\sqrt{2}}.$$

Also, $CS = \sqrt{\alpha^2 - \beta^2} = \sqrt{10 - 2} = 2\sqrt{2}.$

Hence, referred to ∥ axes through C as axes of co-ordinates, the foci are at $(-2, 2)$ and $(2, -2)$.]

**11. Find the latus-rectum, eccentricity and foci of an ellipse in the following cases :—

(i) $x^2 + 5y^2 = 17$.

(ii) $3x^2 + 7y^2 - 28x + 10y = 6$.

(iii) $3x^2 + 4xy + 2y^2 + 4x + 12 = 0$.

**12. Trace the ellipse of Ex. 8.

[*Hints.* (i) Find the centre, and the lengths and position of the axes. This gives the points A, A', B, B' on the curve.

(ii) Find the co-ordinates of the foci, and also the latus-rectum. This helps us in finding the points L, L', L_1, L'_1 at the ends of the *latera recta*.

(iii) Find the points P, P', Q, Q' where the curve cuts the old axes.

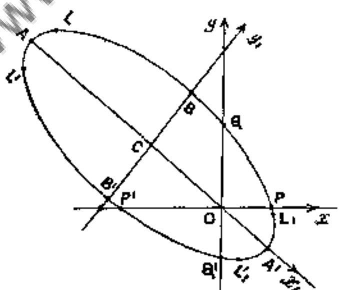

FIG. 40

(iv) A knowledge of the *nature* of the curve will now help us in tracing the curve fairly well (fig. 40). [14]]

**13. Trace the ellipses of Ex. 9.

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[14] To ensure greater accuracy, a few more points may be plotted with the help of the original equation of the curve. (See foot-note 64 of Ch. I.)

**14. Show that the foci of the ellipse

$$ax^2+2hxy+by^2=1,\ ab>h^2,$$

are given by

$$\{(ax+hy)^2-(hx+by)^2\}/(a-b)=(ax+hy)(hx+by)/h=S.$$

[*Solution.* It can be easily shown that the equation of the pair of tangents from the focus of a conic to the conic itself satisfies the conditions for a circle.[15]

Now, the equation of the pair of tangents from $P\equiv(x_1, y_1)$ is $SS_1=T^2$ (S, S_1, T being as usual).

Hence, if P be a focus, we must have

$$aS_1-(ax_1+hy_1)^2=bS_1-(hx_1+by_1)^2,$$

and $\quad hS_1=(ax_1+hy_1)(hx_1+by_1).$[16]

$$\therefore\ \{(ax_1+hy_1)^2-(hx_1+by_1)^2\}/(a-b)$$
$$=S_1=(ax_1+hy_1)(hx_1+by_1)/h.$$

Hence the result.]

**15. Find the foci of the conic

$$ax^2+2hxy+by^2+2gx+2fy+c=0.$$

---

[15] For this purpose, let the focus P of the conic be taken as the origin, and let the x-axis be perpendicular to the corresponding directrix. Then, the equation of the directrix may be written as $x+k=0$. Hence, the equation of the conic is $x^2+y^2=e^2(x+k)^2$, where e= its eccentricity. Now, by Ex. 3 (iii) of Art. 18, the equation of the pair of tangents from $P\equiv(0, 0)$ is

$$-e^2k^2\{x^2(1-e^2)+y^2-2e^2kx-e^2k^2\}=e^4k^2(x+k)^2.$$

$$\therefore\qquad x^2+y^2=0.$$

Hence, the co-efficients of x^2 and y^2 are equal, and the coefficient of xy is zero. Hence the above result.

[16] The first of these equations expresses the condition that the co-efficients of x^2 and y^2 may be equal; while the second states that the co-efficient of xy is zero. (See Art. 17·31.)

38.

ON TANGENTS.

Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Then, by using the methods of Art. 18 (Ch. III), the following results can be easily obtained :—

(i) The equation of the tangent at $P\equiv(x_1, y_1)$ is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1.$$ [17]

(ii) The tangent at the point 'ϕ' is

$$\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi=1.$$ [18]

(iii) The condition that the line $y=mx+c$ may touch the ellipse is $c^2=a^2m^2+b^2$. [19-20]

(iv) The equation of the pair of tangents from (x_1, y_1) is

$$SS_1=T^2.$$ [21]

where $S\equiv\frac{x^2}{a^2}+\frac{y^2}{b^2}-1$, $S_1=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$, and

$$T\equiv\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1.$$

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[17] See Ex. 3 (ii) of Art. 18. Students must have already worked out this result by themselves.

[18] Putting $x_1=a\cos\phi$ and $y_1=b\sin\phi$ in the previous equation, we have this result.

[19] See Ex. 8 (ii) of Art. 18. The lines $y=mx\pm\sqrt{a^2m^2+b^2}$ are, therefore, both tangents to the ellipse for all values of m.

[20] As in Art. 18·4, it can be easily seen with the help of this equation that two tangents can be drawn to an ellipse from any point P; and that these tangents will be *real*, *coincident*, or *imaginary* according as P is *within*, *on*, or *without* the ellipse.

[21] To obtain this result, proceed exactly as in Art. 18·5.

38·1. A geometrical theorem.

Tangents at the extremities of any chord through the centre of an ellipse are parallel to each other.

Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$; and let $P\equiv(x_1, y_1)$ be one of the extremities of any chord through the centre. Then, its other extremity is $Q\equiv(-x_1, -y_1)$

Now, the tangents at P and Q are

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1, \text{ and } \frac{xx_1}{a^2}+\frac{yy_1}{b^2}=-1 \text{ respectively.}$$

The 'm' of each of these lines being $=-\frac{b^2x_1}{a^2y_1}$, the tangents at P and Q are parallel to each other.

Examples.

1. Write down the equation of the tangent at (3, 4) to the ellipse $5x^2+y^2=7$.

2. Show that the tangents at the points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ whose eccentric angles are θ and ϕ intersect at the point given by

$$x=\frac{a\cos\frac{1}{2}(\theta+\phi)}{\cos\frac{1}{2}(\theta-\phi)} \text{ and } y=\frac{b\sin\frac{1}{2}(\theta+\phi)}{\cos\frac{1}{2}(\theta-\phi)}.$$

3. A and A' are the extremities of the major axis of an ellipse, and the tangent at any point P on the ellipse meets the tangent at A in Q; show that the line joining the centre C of the ellipse to Q is $\parallel$ to $A'P$.

4. Show that the condition that the line $x \cos \alpha + y \sin \alpha - p = 0$ may touch the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $p^2 = a^2 \cos^2 \alpha + b^2 \sin^2 \alpha$.

5. Find the angle between the tangents drawn from $P \equiv (\alpha, \beta)$ to the ellipse of Ex. 4.

6. If ϕ be the angle between the tangents from P to the ellipse of Ex. 4, and the points, S C, S', are as in fig. 38, then $2\, SP.\, S'P \cos \phi = SP^2 + S'P^2 - 4a^2$.

**7. Find the equation of the pair of tangents from (x_1, y_1) to the ellipse $ax^2 + 2hxy + by^2 = 1$, $(ab > h^2)$.

39.

ON CHORDS.

Let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Then, by using the methods of Art. 19 (Ch.III), the following results can be easily obtained :—

(i) The equation of the chord joining the points (x_1, y_1) and (x_2, y_2) is

$$\frac{(x - x_1)(x - x_2)}{a^2} + \frac{(y - y_1)(y - y_2)}{b^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1.$$ [22]

(ii) The equation of the chord whose middle point is at (x_1, y_1) is $T = S_1$, [23]
where S_1 and T are as in Art. 38.

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[22] Consider this equation, and proceed as in Art. 19·1.

[23] See Ex. 4 of Art. 19 where it has been worked out.

(iii) The equation of the chord of contact of tangents to the ellipse from the point (x_1, y_1) outside it is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1.$$ [24]

39·1. To find the equation of the chord joining the points 'θ' and 'ϕ' on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

The equation of the chord is evidently

$$\frac{x-a\cos\theta}{a(\cos\theta-\cos\phi)}=\frac{y-b\sin\theta}{b(\sin\theta-\sin\phi)}.$$

$$\therefore \quad \frac{x-a\cos\theta}{a\sin\frac{1}{2}(\phi-\theta)\sin\frac{1}{2}(\phi+\theta)}=\frac{y-b\sin\theta}{b\sin\frac{1}{2}(\theta-\phi)\cos\frac{1}{2}(\theta+\phi)}.$$

$$\therefore \frac{x}{a}\cos\frac{\theta+\phi}{2}+\frac{y}{b}\sin\frac{\theta+\phi}{2}=\cos\frac{\theta+\phi}{2}\cos\theta$$

$$+\sin\frac{\theta+\phi}{2}\sin\theta=\cos\frac{\theta-\phi}{2}, \quad \ldots \quad \ldots \quad (i)$$

which is a simplified form of the equation of the chord.

39·11. Note on the above.

From (i), it is clear that if the sum of the eccentric angles of any two points on the ellipse is a constant ($=2\lambda$ say), then the chord joining the two points is always parallel to the fixed straight line $\frac{x}{a}\cos\lambda+\frac{y}{b}\sin\lambda=1$; and hence, is parallel to the tangent at the point whose eccentric angle is λ. [25]

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[24] Proceed as in Art. 19·3.

[25] Conversely, if the chord joining any two points on the ellipse is parallel to a fixed line, then the sum of the eccentric angles of the two points = a constant.

Examples.

1. Find the equation of the chord of contact of tangents from (x_1, y_1) to the curve $ax^2 + 2hxy + by^2 = 1$.

2. The chord joining any two points on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ meets the major axis at the same point. If α and β be the eccentric angles of such a pair of points, then $\tan \frac{1}{2}\alpha \tan \frac{1}{2}\beta$ = a constant.

40.

ON NORMALS.

Let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Then, by using the method of Art. 20, the following results can be easily obtained :—

(i) The normal at any point (x_1, y_1) on the ellipse is

$$\left(x - x_1\right) \bigg/ \frac{x_1}{a^2} = \left(y - y_1\right) \bigg/ \frac{y_1}{b^2}.[26] \quad \ldots \quad (1)$$

(ii) The equation of the normal at the point 'ϕ' is

$$\frac{ax}{\cos\phi} - \frac{by}{\sin\phi} = a^2 - b^2.[27] \quad \ldots \quad (2)$$

40·1. On the number of normals from a given point.

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[26] Proceed as in Art. 20·2.

[27] Putting $x_1 = a\cos\phi$, $y_1 = b\sin\phi$ in (1), we have

$$(x - a\cos\phi)\frac{a}{\cos\phi} = (y - b\sin\phi)\frac{b}{\sin\phi}.$$

Re-arranging this, we have (2).

Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, and let $P\equiv(h, k)$ be the given point.

The normal at 'ϕ' is $\frac{ax}{\cos\phi}-\frac{by}{\sin\phi}=a^2-b^2$.

If it passes through P, we must have

$$\frac{ah}{\cos\phi}-\frac{bk}{\sin\phi}=a^2-b^2. \quad \ldots \quad \ldots \quad (3)$$

Now, putting $\tan\phi/2=t$, we have

$$\cos\phi=\frac{1-t^2}{1+t^2} \text{ and } \sin\phi=\frac{2t}{1+t^2}.$$

And substituting these values of $\cos\phi$ and $\sin\phi$ in (3) and simplifying, we have

$$bkt^4+2t^3(ah+a^2-b^2)+2t(ah-a^2+b^2)-bk=0, \quad \ldots \quad (4)$$

a biquadratic equation in t.

Hence, it is clear that *four normals can generally be drawn to an ellipse from any point (in its plane).* [28]

**40·2. Some theorems.

(i) *If the normals at four points on an ellipse meet in a point, then the sum of the eccentric angles of the four points is equal to an odd multiple of π.*

Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$; and

Let the normals at 'α', 'β', 'γ' and 'δ' meet at $P\equiv(h, k)$.

Then, by Art. 40·1, the roots of (4) in t are $\tan\frac{1}{2}\alpha$, $\tan\frac{1}{2}\beta$, $\tan\frac{1}{2}\gamma$ and $\tan\frac{1}{2}\delta$.

---

[28] Each value of t gives one point (real or imaginary) on the ellipse.

$$\therefore \quad \Sigma \tan \tfrac{1}{2}\alpha = -2(ah + a^2 - b^2)/bk,$$
$$\Sigma \tan \tfrac{1}{2}\alpha \tan \tfrac{1}{2}\beta = 0,$$
$\Sigma \tan \frac{1}{2}\alpha \tan \frac{1}{2}\beta \tan \frac{1}{2}\gamma = -2(ah - a^2 + b^2)/bk$ and
$\tan \frac{1}{2}\alpha \tan \frac{1}{2}\beta \tan \frac{1}{2}\gamma \tan \frac{1}{2}\delta = -1.$

$$\therefore \quad \tan \tfrac{1}{2}(\alpha + \beta + \gamma + \delta) = \frac{4(b^2 - a^2)}{0} = \infty.$$

$$\therefore \quad \frac{\alpha + \beta + \gamma + \delta}{2} = \text{an odd multiple of } \frac{\pi}{2}.$$

$$\therefore \quad \alpha + \beta + \gamma + \delta = \text{an odd multiple of } \pi.$$

(ii) *If the normals at the points 'α', 'β' and 'γ' on an ellipse are concurrent, then* $\Sigma \sin(\alpha + \beta) = 0$.

Let $P \equiv (h, k)$ be the point where the normals meet, and let 'δ' be the fourth point on the ellipse the normal at which passes through P.

Then, by (i), we have

$$\Sigma \tan \tfrac{1}{2}\alpha \tan \tfrac{1}{2}\beta = 0$$
$$\text{and } \tan \tfrac{1}{2}\alpha \tan \tfrac{1}{2}\beta \tan \tfrac{1}{2}\gamma \tan \tfrac{1}{2}\delta = -1.$$

Eliminating $\tan \frac{1}{2}\delta$ from these equations, we have

$$\Sigma(\tan \tfrac{1}{2}\alpha \tan \tfrac{1}{2}\beta - \cot \tfrac{1}{2}\alpha \cot \tfrac{1}{2}\beta) = 0.$$

$$\therefore \Sigma \frac{\sin^2 \frac{1}{2}\alpha \sin^2 \frac{1}{2}\beta - \cos^2 \frac{1}{2}\alpha \cos^2 \frac{1}{2}\beta}{\sin \frac{1}{2}\alpha \cos \frac{1}{2}\alpha \sin \frac{1}{2}\beta \cos \frac{1}{2}\beta} = 0.$$

$$\therefore \Sigma \quad \frac{\cos \frac{1}{2}(\alpha + \beta) \cos \frac{1}{2}(\alpha - \beta)}{\sin \alpha \sin \beta} = 0,$$ by factorizing the numerator.

$$\therefore \Sigma \quad \sin \gamma (\cos \alpha + \cos \beta) = 0.$$

$$\therefore \Sigma \quad \sin(\alpha + \beta) = 0.$$

Examples.

1. Show that the condition that the line $lx+my+n=0$ may be a normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$$\frac{a^2}{l^2}+\frac{b^2}{m^2}=\frac{(a^2-b^2)^2}{n^2}.$$

[*Hints.* The line must be the same as $\frac{ax}{\cos\phi}-\frac{by}{\sin\phi}=a^2-b^2$. Hence, $\frac{l\cos\phi}{a}=\frac{m\sin\phi}{-b}=\frac{n}{b^2-a^2}$. Now, eliminate ϕ from these two equations.]

2. Find the condition that the line $x\cos\alpha+y\sin\alpha-p=0$ may be a normal to the ellipse $ax^2+by^2=1$.

3. If P' be the point on the auxiliary circle corresponding to the point P on an ellipse, show that the normals at P and P' meet on a fixed circle.

4. If normals be drawn at the extremities of a focal chord of an ellipse, then the line through their point of intersection parallel to the major axis will bisect the chord.

5. Find the lengths of the *subtangent* and the *subnormal* for the ellipse of Ex. 1. [29]

**7. Show that the feet of the normals from the point (h, k) to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ lie on the conic

$$(a^2-b^2)xy+b^2kx-a^2hy=0.$$

[29] The terms are defined as in the case of the parabola, the major axis of the ellipse taking the place of the axis of the parabola.

[*Solution.* The normal at (x_1, y_1) on the ellipse is $(x-x_1)\Big/\frac{x_1}{a^2}=(y-y_1)\Big/\frac{y_1}{b^2}$. It passes through (h, k), if

$$(h-x_1)\Big/\frac{x_1}{a^2}=(k-y_1)\Big/\frac{y_1}{b^2}.$$

Hence, the feet of the normals lie on the conic $(h-x)\Big/\frac{x}{a^2}=(k-y)\Big/\frac{y}{b^2}$. Hence the result.]

**8. Show that there are eight normals to the ellipse $ax^2+by^2=1$ which touch the ellipse $a'x^2+b'y^2=1$.

**9. If the normals at the points on the ellipse of Ex. 1 whose eccentric angles are α, β, γ and δ meet in a point, then $\Sigma \cos \alpha \, \Sigma \sec \alpha = 4$.

**10. Show that if the normals at the points where the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is cut by the lines $lx+my=1$ and $l'x+m'y=1$ meet in a point, then we must have $a^2 ll'=b^2 mm'=-1$.

[*Solution.* The equation of any conic through the points of intersection of the ellipse and the given lines is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\lambda(lx+my-1)(l'x+m'y-1)=0.$$

Now, let the normals at the points of intersection meet at the point (h, k). Then, by Ex. 7, the above equation must be the same as $(a^2-b^2)xy+b^2kx-a^2hy=0$ for some value of λ.

Hence, comparing the two equations, we have

$$\frac{1}{a^2}-\lambda ll'=0,\ \frac{1}{b^2}-\lambda mm'=0 \text{ and } 1+\lambda=0.$$

$\therefore \quad a^2\, ll' = b^2\, mm' = -1.$]

**11. If the normals at four points on an ellipse meet in a point, find the equations of the two parabolas passing through the four points.

[*Hints.* Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$;

and let the normals meet at (h, k). Then, by Ex. 7, any conic through the feet of the normals is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1+\lambda\left\{(a^2-b^2)xy+b^2kx-a^2hy\right\}=0.$$

Now, express the condition that this may be a parabola, find the values of λ from it, and finally substitute its value in the above equation.]

41.

ON POLES AND POLARS.

Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Then, by using the methods of Art. 21, the following results can be easily proved :—

(i) The polar of $P\equiv(x_1, y_1)$ is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$. [30]

(ii) The pole of the line $y=mx+c$ is $\left(-\frac{ma^2}{c}, \frac{b^2}{c}\right)$. [31]

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[30] See Ex. 2 (2) of Art. 21.

[31] See Ex. 6 (2) of Art. 21.

(iii) If the polar of P passes through Q, then the polar Q will pass through P.[32]

Examples.

1. Find the pole of the line $3x+4y=7$ with respect to the ellipse $3x^2+y^2=2$.

2. *Show that the focus of an ellipse is the pole of the corresponding directrix.*

3. If the pole of the normal at P on an ellipse lie on the normal at Q, then the pole of the normal at Q will lie on the normal at P.

**4. If the polar of P with respect to a conic represented by the general equation of the second degree in x, y passes through Q, then the polar of Q will pass through P.

5. If CQ be conjugate to the normal at P on an ellipse, then CP will be conjugate to the normal at Q.

6. Show that the condition that the lines $y=mx+c$ and $y=m'x+c'$ may be conjugate with respect to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $b^2+a^2mm'=cc'$.

[*Hints.* Proceeding as in Art. 21·4, we find that the condition that the pole of the first line may lie on the second is $b^2+a^2mm'=cc'$. By the symmetry of this result, it is clear that the pole of the 2nd. line will also lie on the first. Hence the result.]

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[32] Proceed as in Art. 21·4 (or find the condition that the pole of $y=mx+c$ may lie on $y=m'x+c'$, and note the *symmetry* of the result thus obtained).

42.

ON DIAMETERS.

42·1. On the locus of the middle points of a system of parallel chords of an ellipse.

Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$; and let $P\equiv(x_1, y_1)$ be any point on the locus. Now, the equation of the chord having $P\equiv(x_1, y_1)$ as its middle point is

$$T=S_1,$$

$$\text{or}\quad \frac{xx_1}{a^2}+\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}.$$

If it is parallel to the line $y=mx$, we must have

$$m=-\frac{b^2x_1}{a^2y_1}.$$

Hence, the locus of P is given by

$$ma^2y+b^2x=0.\text{ [33]}$$

It is a straight line passing through the centre of the ellipse, and is called a *diameter* of the ellipse (see footnote 35 of Ch. V).

42·11. Definitions.

And chord bisected by a diameter is called the *double ordinate* of that diameter.

Two diameters are said to be *conjugate* to each other when each bisects chords parallel to the other. Two conjugate diameters whose lengths are equal are called *equi-conjugate diameters*.

[33] See Art. 34·1. Note that the method is exactly the same as that of Art. 34·1 [or 23·1 (II)].

42·2. Find the condition that the lines $y=mx$ and $y=m'x$ may be conjugate diameters of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

By Art. 42·1, the equation of the diameter which bisects all chords parallel to the line $y=mx$ is

$$ma^2y+b^2x=0. \quad \ldots \quad \ldots \quad (1)$$

Hence, if $y=m'x$ be the diameter conjugate to $y=mx$, it must be the same as (1).

$$\therefore \quad m'=-\frac{b^2}{a^2m},$$

$$\text{or} \quad mm'=-\frac{b^2}{a^2}. \quad \ldots \quad \ldots \quad (2)$$

This is, therefore, the condition of *conjugacy.*

42·21. Conjugate diameters and conjugate lines.

In the special case when $c=c'=0$, the equations of the lines of Ex. 6 of Art. 41 become $y=mx$ and $y=m'x$, and the condition of conjugacy reduces to

$$b^2+a^2mm'=0.$$

But this is also the condition that the lines may be a pair of conjugate diameters of the ellipse. Hence, a pair of conjugate diameters may be regarded as a special case of a pair of conjugate lines.

42·22. If θ and ϕ be the eccentric angles of the extremities of two conjugate semi-diameters of an ellipse, then

$$\theta \sim \phi = \textbf{an odd multiple of } \frac{\pi}{2}.$$

Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$; and let CP and CD be a pair of conjugate semi-diameters such that $P\equiv(a\cos\theta, b\sin\theta)$ and $D\equiv(a\cos\phi, a\sin\phi)$.

Now, the 'm' of the line CP is $\frac{b}{a}\tan\theta$ (fig. 39); and that of the line CD is $\frac{b}{a}\tan\phi$.

And since the lines CP and CD lie along a pair of conjugate diameters, we must have

$$\frac{b^2}{a^2}\tan\theta\tan\phi=-\frac{b^2}{a^2}, \text{ by (2).}$$

$$\therefore \quad \tan\theta\tan\phi+1=0.$$

$$\therefore \quad \sin\theta\sin\theta+\cos\theta\cos\phi=0,$$

or, $$\cos(\theta-\phi)=0.$$

$$\therefore \quad \theta\sim\phi=\text{an odd multiple of } \frac{\pi}{2}.$$ [34]

42·3. Some geometrical theorems.

(i) *The tangent at the extremity of a diameter is parallel to the conjugate diameter.*

Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$; and let $P\equiv(a\cos\theta, b\sin\theta)$ be any point on it.

---

[34] Representing the state of affairs by means of a diagram, it can be easily seen that θ and $\theta+\frac{\pi}{2}$ may be taken to be the eccentric angles of the extremities of two conjugate semi-diameters.

Now, since the 'm' of CP is $\frac{b}{a}\tan\theta$, that of the diameter conjugate to it is $-\frac{b}{a}\cot\theta$.

Again, since the equation of the tangent at P is $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$, the '$m$' of the tangent is $-\frac{b}{a}\cot\theta$.

Combining these two results, we have the above theorem.[35]

(ii) *The chords joining any point on an ellipse to the extremities of any diameter are parallel to a pair of conjugate diameters.*[36]

Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$; and let $P\equiv(a\cos\theta, b\sin\theta)$ be any point on it. Then, the other extremity of the diameter CP is $P'\equiv(-a\cos\theta, -b\sin\theta)$.

Now, if $Q\equiv(a\cos\alpha, b\sin\alpha)$ be any other point on the ellipse, the 'm' of the line PQ is

$$\frac{b(\sin\alpha-\sin\theta)}{a(\cos\alpha-\cos\theta)};\text{ and that of the line } P'Q \text{ is}$$

$$\frac{b(\sin\alpha+\sin\theta)}{a(\cos\alpha+\cos\theta)}.$$

$$\text{Their product }=\frac{b^2}{a^2}\cdot\frac{\sin^2\alpha-\sin^2\theta}{\cos^2\alpha-\cos^2\theta}=\frac{b^2}{a^2}\cdot\frac{\sin^2\alpha-\sin^2\theta}{\sin^2\theta-\sin^2\alpha}.$$

$$=-\frac{b^2}{a^2}.$$

---

[35] Hence, if CD be the semi-diameter parallel to the tangent at P, CP and CD form a pair of conjugate semi-diameters.

[36] Two such chords are said to form a pair of *supplemental chords.*

Hence, by (2), PQ and $P'Q$ are parallel to a pair of conjugate diameters. Hence the theorem.[37]

(iii) The sum of the squares of two conjugate semi-diameters of an ellipse is constant.

Let the equation of the ellipse be $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$. Also let θ and $\theta+\dfrac{\pi}{2}$ be the eccentric angles of the extremities of two conjugate semi-diameters (see foot-note 34 of this Ch.), and let them be denoted by P and D respectively.

Then, $P\equiv(a\cos\theta,\ b\sin\theta)$ and $D\equiv(-a\sin\theta,\ b\cos\theta)$.

$\therefore \qquad CP^2 = a^2\cos^2\theta + b^2\sin^2\theta.$

and $\qquad CD^2 = a^2\sin^2\theta + b^2\cos^2\theta.$

$\therefore \qquad CP^2 + CD^2 = a^2 + b^2 =$ a constant.

(iv) If P and D be the extremities of two conjugate semi-diameters, then the $\triangle PCD$ is of constant area. And if P' and D' be the other extremities of these diameters, then the parallelogram formed by the tangents at P, P' D and D' is also of constant area.

Let the equation of the ellipse be $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

And let the points P and D be so taken that ϕ and $\phi+\dfrac{\pi}{2}$ may be taken to be the eccentric angles of these points.

Then, $P\equiv(a\cos\phi,\ b\sin\phi)$ and $D\equiv(-a\sin\phi,\ b\cos\phi)$.

$$\therefore\quad \triangle\ PCD = \tfrac{1}{2}\begin{vmatrix} 0, & 0, & 1 \\ a\cos\phi, & b\sin\phi, & 1 \\ -a\sin\phi, & b\cos\phi, & 1 \end{vmatrix} = \tfrac{1}{2}ab.$$

---

[37] Hence, we have a *mechanical* method of drawing pairs of conjugate diameters of an ellipse by (i) taking any point Q on the ellipse, (ii) joining it to the extremities P and P' of any diameter PCP', and then (iii) drawing through C the diameters parallel to PQ and $P'Q$.

Again, representing the state of affairs by means of a diagram, it can be easily seen that the area of the parallelogram = 8. $\triangle FCD = 4\,ab$.
Hence the above results.

42·4. To find the angle between a pair of conjugate diameters.

Let a_1, b_1 be the lengths of two conjugate semi-diameters CP and CD; and let ω be the angle between them. Then,

$$\triangle PCD = \tfrac{1}{2}\, a_1 b_1 \sin\omega.$$
$$= \tfrac{1}{2}\, ab, \text{ by (iv) of Art. 42·3.}$$

$$\therefore \quad a_1\, b_1 \sin\omega = ab.$$

$$\therefore \qquad \omega = \sin^{-1}\frac{ab}{a_1 b_1}.$$

***42·5. To find the equation of an ellipse with reference to a pair of conjugate diameters as axes.**

Let the equation of the ellipse be $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ ... (3)

with reference to its axes as axes of co-ordinates.

Also, let the axes be turned so that they may coincide with a pair conjugate semi-diameters CP and CD whose lengths are a_1 and b_1 respectively. Then, since the origin is unchanged, (3) is transformed into an equation of the form

$$Ax_1^2 + 2Hx_1y_1 + By_1^2 = 1. \qquad \dots \quad (4)$$

Now, since $P \equiv (a_1, 0)$ and $D \equiv (0, b_1)$ are on (4), we must have $Aa_1^2 = 1$ and $Bb_1^2 = 1$.

$$\therefore \qquad A = \frac{1}{a_1^2} \text{ and } B = \frac{1}{b_1^2}.$$

Again, since CP bisects all chords parallel to CD, we must have two equal and opposite values of y_1 for every value of x_1. From (4), it is clear that this cannot be unless the co-efficient of y_1 is zero. Hence, $H=0$.

Hence, the equation (4) reduces to

$$\frac{x_1^2}{a_1^2}+\frac{y_1^2}{b_1^2}=1. \quad \ldots \quad \ldots (5)$$

42·6. On the equi-conjugate diameters of an ellipse.

Let the equation of the ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

and let P and D be the extremities of two conjugate semi-diameters whose eccentric angles are and ϕ and $\phi+\pi/2$ respectively. Then,

$CP^2=a^2 \cos^2 \phi+b^2 \sin^2 \phi$ and $CD^2=a^2 \sin^2 \phi + b^2\cos^2 \phi$.

Now, if $CP=CD$, we have

$$\cos^2\phi\,(a^2-b^2)=\sin^2\phi\,(a^2-b^2).$$

$$\therefore \quad \cos^2 \phi=\sin^2\phi,$$

$$\text{or } \cos 2\phi=0.$$

$$\therefore \quad \phi=\text{an odd multiple of } \frac{\pi}{4}.$$

Now, representing the state of affairs by means of a diagram, it can be easily seen that $\frac{\pi}{4}$ and $\frac{3\pi}{4}$ are the *only* values of ϕ which admit of a new diameter in each case. Hence, there is *only one* pair of equi-conjugate diameters of an ellipse.

From the above it, it clear that the equations of the equi-conjugate diameters are $y = \pm \frac{b}{a} x$.

Hence, they are equally inclined to either axis of the ellipse.

Examples.

1. Show that the tangents at the ends of any chord of an ellipse intersect on the diameter which bisects the chord.

2. The product of the focal distances of any point P on the ellipse is equal to the square of the semi-diameter conjugate to that to the point P.

[*Solution.* Let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$; and let P and D be the extremities of two conjugate semi-diameters whose eccentric angles are ϕ and $\phi + \frac{\pi}{2}$ respectively. Then,

$$CD^2 = a^2 \sin^2 \phi + b^2 \cos^2 \phi.$$

Now, $S \equiv (-ae, 0)$ and $S' \equiv (ae, 0)$.

$$\therefore \quad SP^2 = a^2(e + \cos \phi)^2 + b^2 \sin^2\phi.$$
$$= a^2 + a^2 e^2 \cos^2\phi + 2a^2 e \cos \phi.$$

$$\therefore \quad SP = a + ae \cos \phi.$$

Similarly, $S'P = a - ae \cos \phi$.

$$\therefore \quad SP . S'P = a^2 - a^2 e^2 \cos^2 \phi - a^2 \sin^2\phi + b^2 \cos^2\phi$$
$$= CD^2].$$

3. QVQ' is a double ordinate of CP; and the tangent at Q meets CP in R; show that $CV . CR = CP^2$.

4. If P and D be the extremities of conjugate diameters of an ellipse, then the tangents at P and D meet on another ellipse.

5. Show that the angle which a diameter of an ellipse subtends at A is supplementary to the angle subtended at B by the diameter conjugate to the former.

**6. The line joining the extremities of any two semi-diameters of an ellipse which are at right angles to each other will always touch a fixed circle.

[*Solution.* A variable line will touch a fixed circle, if its perpendicular distance from the centre of the circle is equal to the radius of the circle.

Now, let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$;

and let $x \cos \alpha + y \sin \alpha - p = 0$ be the line joining the extremities P, Q of any two semi-diameters of the ellipse. Then, the equation of the diameters is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\left(\frac{x \cos\alpha + y \sin \alpha}{p}\right)^2,$$

since they are the lines joining the origin to the points of intersection of the ellipse and PQ.

These diameters are perpendicular to each other, if

$$\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{p^2}.$$

Hence, p = a constant. Hence, the perpendicular distance of PQ from the fixed point C being a constant, PQ touches a fixed circle.]

7. QQ' is any chord of an ellipse parallel to one of its equi-conjugate diameters, and the tangents at Q and

Q' meet in P; show that the circle QPQ' passes through the centre of the ellipse.

**8. Show that the straight lines joining the centre of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ to the points where the line $y=mx+\sqrt{\frac{a^2m^2+b^2}{2}}$ cuts it are conjugate diameters.

**9. Any tangent to an ellipse meets the director circle in E and F. Show that CE, CF lie along conjugate diameters of the ellipse.

**10. Show that the chord joining the extremities of a pair of conjugate semi-diameters of an ellipse touches a similar ellipse.

**11. Show that the length of the perpendicular from the centre of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ to the tangent at $P\equiv(x_1, y_1)$ on it $=\frac{ab}{b_1}$, where b_1 = semi-diameter conjugate so that to the point P.

**12. *Show that the length of the chord joining the points 'α', 'β' on the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *is equal to* $2b_1\sin\frac{\alpha-\beta}{2}$, *where* b_1 *is the semi-diameter parallel to the given chord*

[*Solution.* $(\text{chord})^2=a^2(\cos\alpha-\cos\beta)^2+b^2(\sin\alpha-\sin\beta)^2$.

$$=4a^2\sin^2\frac{\beta+\alpha}{2}.\sin^2\frac{\alpha-\beta}{2}+4b^2\sin^2\frac{\alpha-\beta}{2}.\cos^2\frac{\alpha+\beta}{2}$$

$$= 4\sin^2\frac{\alpha-\beta}{2}\left\{a^2\sin^2\frac{\alpha+\beta}{2}+b^2\cos^2\frac{\alpha+\beta}{2}\right\}$$

$$= 4\sin^2\frac{\alpha-\beta}{2}.\,k^2,$$

where k = the semi-diameter conjugate to the diameter to the point $\frac{\alpha+\beta}{2}$. Now, it can be easily seen that the diameter conjugate to that to the point $\frac{\alpha+\beta}{2}$ is parallel to the given chord. Hence, $k = b_1$, by hypothesis.

$$\therefore \quad \text{chord} = 2b_1 \sin\frac{\alpha-\beta}{2}.]$$

13. Show that the length of the chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ which touches the ellipse $\frac{x^2}{a^2-\lambda^2}+\frac{y^2}{b^2-\lambda^2}=1$ is $\frac{2\lambda b_1^2}{ab}$, where b_1 = the semi-diameter ∥ to the chord.

**14. The normals at the point P, Q, R, T on an ellipse meet in a point, and the circle through P, Q, R cuts the conic again in T'; show that TT' is a diameter of the ellipse.

[*Hints.* Use Ex. 6 of Art. 37 and Art. 40·2 (i) in showing that the co-ordinates of T' are minus those of T.]

42.

ON SOME LOCUS PROBLEMS CONNECTED WITH THE ELLIPSE.

42·1. The direct method.

To find the locus of points the tangents from which to an ellipse are at right angles to each other.[38]

[38] See Ex. 2 of Art. 23.

Let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

By proceeding exactly as in Art. 23·1 (I), it can be easily seen that the equation of the locus is

$$x^2 + y^2 = a^2 + b^2.$$

[it is a circle, and is called the *director circle* of the ellipse.] [39]

43·2. The parametric method.

(I) *To find the locus of the middle points of chords of an ellipse which are of constant length.* [40]

Let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$; also

let $P \equiv (a \cos \theta, b \sin \theta)$ and $Q \equiv (a \cos \phi, b \sin \phi)$ be any two points on it. Then, the middle point (x, y) of PQ is given by

$$\left.\begin{aligned} x &= \frac{a}{2}(\cos \theta + \cos \phi), \\ y &= \frac{b}{2}(\sin \theta + \sin \phi). \end{aligned}\right\} \quad \ldots \quad \ldots \quad (1)$$

And since PQ is of constant length, we have

$$a^2(\cos \theta - \cos \phi)^2 + b^2(\sin \theta - \sin \phi)^2 = k^2 \text{ (say)} \ldots \quad (2)$$

Now, to find the equation of the locus, we must eliminate θ and ϕ from (1) and (2).

---

[39] In case of the parabola $y^2 = 4ax$, the corresponding locus is given by $x + a = 0$, which is the equation of its directrix. And as the parabola may be regarded as a limiting case of the ellipse, *the directrix of the parabola may be regarded as a limiting case of the director circle of the ellipse.*

[40] Note that Ex. 8 of Art. 23 is a special case of this.

For this purpose, we have

$$\left.\begin{aligned} x &= a\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2} \\ y &= b\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2} \end{aligned}\right\}, \text{ from } (1).$$

$$\therefore \tan\frac{\theta+\phi}{2} = \frac{ay}{bx}. \quad \ldots \quad \ldots \quad (3)$$

by division ; and $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2\frac{\theta-\phi}{2}$. ... ... (4)

by squaring and adding. Again, from (2), we have

$$k^2 = 4\sin^2\frac{\theta-\phi}{2}\left\{a^2\sin^2\frac{\theta+\phi}{2} + b^2\cos^2\frac{\theta+\phi}{2}\right\}.$$

$$\therefore k^2 = 4\left\{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right\}\left\{\frac{a^4y^2 + b^4x^2}{a^2y^2 + b^2x^2}\right\}, \text{by (3) and (4).}$$

Hence, the equation of the locus is given by

$$k^2a^2b^2(a^2y^2 + b^2x^2) = 4(a^2b^2 - b^2x^2 - a^2y^2)(a^4y^2 + b^4x^2).$$

(II) *To find the locus of the point of intersection of normals at the extremities of conjugate semi-diameters of an ellipse.*

Let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$; also let θ and $\theta + \frac{\pi}{2}$ be the eccentric angles of the extremities of a pair of conjugate semi-diameters.

Now, the equation of the normals at these points are

$$\left.\begin{aligned} \frac{ax}{\cos\theta} - \frac{by}{\sin\theta} &= a^2 - b^2 \\ \text{and} \quad \frac{ax}{\sin\theta} + \frac{by}{\cos\theta} &= b^2 - a^2. \end{aligned}\right\} \quad \ldots \quad \ldots \quad (5)$$

And since these equations hold together at the point of intersection of the normals, the equation of the locus is obtained by eliminating θ from them.

For this purpose, adding the equations (5) and simplifying, we have

$$\frac{ax+by}{\cos\theta}+\frac{ax-by}{\sin\theta}=0.$$

$$\therefore \quad \frac{\sin\theta}{by-ax}=\frac{\cos\theta}{ax+by}=\frac{1}{\sqrt{2(b^2y^2+a^2x^2)}}=k \text{ (say)}.$$

Then, substituting in any equation of (5) the values of $\sin\theta$ and $\cos\theta$ as obtained from these equations, we haveo

$$\frac{ax}{ax+by}-\frac{by}{by-ax}=k(a^2-b^2).$$

$$\therefore \quad 2(a^2x^2+b^2y^2)^3=(a^2-b^2)^2(a^2x^2-b^2y^2)^2.$$

This is, therefore, the equation of the locus.

(III) *To find the locus of the point of intersection of the normals to an ellipse which are at right angles to each other.*

The normals to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the points whose eccentric angles are θ and ϕ are

$$\left.\begin{aligned}\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}&=a^2-b^2\\ \text{and}\quad \frac{ax}{\cos\phi}-\frac{by}{\sin\phi}&=a^2-b^2.\end{aligned}\right\} \quad \ldots \quad \ldots \quad (5)$$

If these lines are at right angles to each other, we must have

$$\frac{a^2}{b^2}\tan\theta\tan\phi=-1. \quad \ldots \quad \ldots \quad (7)$$

Now, to obtain the equation of the locus, we must eliminate θ and ϕ from (1) and (2).

For this purpose, we have

$$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = \frac{ax}{\cos\phi} - \frac{by}{\sin\phi}, \text{ by (6).}$$

$$\therefore \quad \frac{ax}{by} \cdot \frac{\cos\phi - \cos\theta}{\sin\phi - \sin\theta} = \cot\theta \cot\phi = -\frac{a^2}{b^2}, \text{ by (7).}$$

$$\therefore \quad \frac{2\sin\dfrac{\theta+\phi}{2}\sin\dfrac{\theta-\phi}{2}}{2\sin\dfrac{\phi-\theta}{2}\cos\dfrac{\phi+\theta}{2}} = -\frac{ay}{bx}.$$

or $$\tan\left(\frac{\theta+\phi}{2}\right) = \frac{ay}{bx}. \quad \ldots \quad \ldots \quad (8)$$

Again, equating the product of the left-hand sides of (6) to the product of the right-hand sides, we have

$$\frac{a^2x^2}{\cos\theta\cos\phi} + \frac{b^2y^2}{\sin\theta\sin\phi} - abxy\left\{\frac{\sin(\theta+\phi)}{\sin\theta\sin\phi\cos\theta\cos\phi}\right\}$$
$$= (a^2 - b^2)^2. \quad \ldots \quad \ldots \quad (9)$$

Moreover, from (7), we have

$$\frac{\sin\theta\sin\phi}{b^2} = \frac{\cos\theta\cos\phi}{-a^2} = \frac{\cos(\theta+\phi)}{-(a^2+b^2)}$$
$$= k \text{ (say).} \quad \ldots \quad \ldots \quad (10)$$

Hence, (9) becomes

$$y^2 - x^2 + \frac{xy\sin(\theta+\phi)}{abk} = (a^2-b^2)^2 k, \quad \text{by (10).}$$

$$\therefore \quad y^2 - x^2 - \frac{2x^2y^2(a^2+b^2)}{b^2x^2 - a^2y^2} = \frac{(a^2-b^2)^2}{a^2+b^2} \cdot \frac{a^2y^2 - b^2x^2}{a^2y^2 + b^2x^2}.$$

$$\therefore \quad (a^2+b^2)(x^2+y^2)(a^2y^2+b^2x^2)^2$$
$$= (a^2-b^2)^2(a^2y^2 - b^2x^2)^2.$$

This is, therefore, the equation of the locus.[41]

**43·3. The method of polar co-ordinates.

To find the locus of the foot of the perpendicular on any tangent to an ellipse from one of its foci.

Let $x\cos\alpha + y\sin\alpha - p = 0$ be any tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Then, we have

$$p^2 = a^2\cos^2\alpha + b^2\sin^2\alpha.\text{ [42]}$$

Also the focus S' is $(ae, 0)$. Hence, if p_1 be the length of the perpendicular from S' to the tangent, we must have

$$p_1 = ae\cos\alpha - \sqrt{a^2\cos^2\alpha + b^2\sin^2\alpha}.$$

This is a relation between the polar co-ordinates (p_1, α) of the foot of the perpendicular with reference to S' as the *pole* and CS' as the *initial line*. Hence, the polar equation of the locus is

$$= ae\cos\theta - \sqrt{a^2\cos^2\theta + b^2\sin^2\theta},$$

$$\text{or } (r - ae\cos\theta)^2 = a^2\cos^2\theta + b^2\sin^2\theta,$$

$$\text{or } r^2 - 2ae\,r\cos\theta = b^2. \qquad \ldots \qquad \ldots \quad (11)$$

43·31. Note on the above.

Changing (11) into Cartesian co-ordinates, we have

$$(x - ae)^2 + y^2 = a^2.$$

Hence, the locus is a circle whose centre is at $(ae, 0)$ and radius $= a$.

Again, changing the origin to the point $(ae, 0)$, we have

$$x^2 + y^2 = a^2,$$

which is the equation of the *auxiliary circle*.

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[41] Note how θ and ϕ (or θ) have been eliminated in these problems.

[42] See Art. 35·3.

Examples.

1. Find the locus of the middle points of chords of an ellipse which pass through a fixed point.

2. Find the locus of the middle points of chords of the circle $x^2+y^2=a^2$ which touch the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

3. *Find the locus of the poles of normal chords of the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

[*Solution.* The normal at any point θ is

$$\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2;$$

and the polar of any point (x_1, y_1) is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1.$$

If these two equations represents the same straight line, we must have

$$\frac{x_1\cos\theta}{a^3}=\frac{y_1\sin\theta}{-b^3}=\frac{1}{a^2-b^2}.$$

$$\therefore \quad \cos\theta=\frac{a^3}{x_1(a^2-b^2)} \text{ and } \sin\theta=\frac{-b^3}{y_1(a^2-b^2)}.$$

Squaring and adding (to eliminate θ), we have

$$\frac{a^6}{x_1{}^2}+\frac{b^6}{y_1{}^3}=(a^2-b^2)^2.$$

Hence, the equation of the locus is

$$\frac{a^6}{x^2}+\frac{b^6}{y^2}=(a^2-b^2)^2.$$ [43]]

[43] See Ex. 3 of Art. 23.

4. Find the locus of the middle points of normal chords of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

5. Find the locus of the middle point of the chord of contact of tangents which cut at right angles.

6. P is any point on an ellipse, and P' is the corresponding point on the auxiliary circle; find the locus of the point of intersection of SP and CP'.

7 The normal at any point P on an ellipse cuts the major axis in G; show that the locus of the middle point of PG is an ellipse.

8. Find the locus of the middle points of chords of an ellipse which subtend a right angle at the centre.

9. If P and D are the extremities of two conjugate diameters of an ellipse, show that the locus of the point of intersection of tangents at P and D is another ellipse, and that the locus of the middle point of PD is a third ellipse.

10. The tangents at P and Q on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ meet at the point T and the lines PS and QS' meet on the curve; show that the locus of T is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}\left(\frac{1-e^2}{1+e^2}\right)^2=1.$$

11. Find the locus of the foot of the perpendicular on any normal to an ellipse from one of its foci.

**12. A point moves in such a manner that the perpendicular from the centre on its polar with respect to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is constant; show that its locus is a coaxial ellipse.

CHAPTER VII

THE HYPERBOLA.

44.

THE HYPERBOLA AND ITS EQUATION.

44·1. Definition and general equation of the hyperbola.

For a hyperbola, $e > 1$. Hence, a hyperbola is the locus of a point which moves so that its distance from a fixed point S bears a constant ratio (which is greater than unity) to its distance from a fixed straight line.[1]

Hence, the general form of the equation of hyperbola is

$$(x - x_1)^2 + (y - y_1)^2 = e^2(x \cos \alpha + y \sin \alpha - p)^2, \quad \ldots \quad (1)$$

where $e > 1$. [Cf. Art. 37·1.]

44·2. Find the standard form of the equation of the hyperbola.

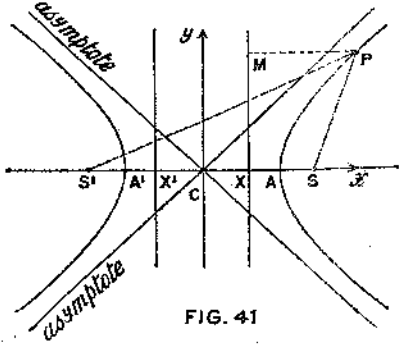

FIG. 41

Let S be the focus, and XM be the directrix of the hyperbola. Draw SX perpendicular to XM, and let $XS = d$ (fig. 41).

[1] As already stated, the fixed point is called the *focus*, and the fixed straight line is called the *directrix* of the hyperbola.

Also let XS and XM be taken as the axes of x and y respectively. Then, the focus S is at $(d, 0)$; and if $P \equiv (x, y)$ be any point on the hyperbola, its equation is

$$(x-d)^2 + y^2 = e^2 x^2.$$

$$\therefore \qquad x^2(e^2-1) + 2dx - y^2 - d^2 = 0,$$

$$\text{or} \quad \left(x + \frac{d}{e^2-1}\right)^2 - \frac{y^2}{e^2-1} = \frac{d^2e^2}{(e^2-1)^2}.$$

Now, changing the origin to the point $C \equiv \left(-\frac{d}{e^2-1}, 0\right)$ without changing the direction of the axes, we have

$$x^2 - \frac{y^2}{e^2-1} = \frac{d^2e^2}{(e^2-1)^2}; \qquad \ldots \qquad \ldots \quad (2)$$

and writing a for $\frac{de}{e^2-1}$ and b^2 for $a^2(e^2-1)$ in (2), we have

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \qquad \ldots \qquad \ldots \quad (3)$$

where a and b are real $(e > 1)$. [2] This is the equation of the hyperbola in what is known as its *standard form*. [Cf. Art. 37·2.]

44·3. On some points connected with the hyperbola.

(i) As the equation (3) is unaltered by writing $-x$ for x and $-y$ for y (jointly or severally), the curve is symmetrical about the axes of x and y, and also about the origin C.

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[2] That the conic whose equation is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (where a and b are real) is a hyperbola can be easily proved by proceeding backwards as in Art. 37·21.

Hence, any chord through C is bisected at the point. [3] The point C is called the *centre* of the hyperbola.

(*ii*) Putting $y=0$ in $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, we have $x=\pm a$.

Hence, the points A and A' where the axis of x cuts the curve are distant a from C, and $AA'=2a$. AA' is called the *transverse axis* of the hyperbola.

Again, putting $x=0$ in (3), we have $y^2=-b^2$. Hence, the curve does not cut the axis of y in any real point. But, if two points B, B' be taken on $x=0$ so that $CB=B'C=b$, then BB' is called the *conjugate axis.*

If N be the foot of the perpendicular from any point P on the hyperbola to the transverse axis, PN is called the *ordinate* of the point P. The double-ordinate through the focus is called the *latus-rectum.*

(iii) From (3), we have $ay=\pm b\sqrt{x^2-a^2}$. Hence, in order that y may be *real*, x must *not* lie between $-a$ and a. Therefore, there is no portion of the curve between A and A'. For any value of x *numerically* greater than a, y will have two equal and opposite values, and will increase indefinitely with x. The curve will, therefore, consist of two branches (one corresponding to positive values of x and the other to negative values of x), and will extend to infinity in both directions. [4]

(iv) From Art 44·2, it is clear that

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[3] See foot-note 64 of chapter I.

[4] From (3), we have $x=\pm a\sqrt{1+\frac{y^2}{b^2}}$. Hence, there are two *real* values of x for every *real* value of y.

$$CX = \frac{d}{e^2-1} = \frac{a}{e} = \frac{CA}{e}.$$

$$\therefore \qquad CA = e.CX. \qquad \ldots \quad (4)$$

Again, proceeding as in Art. 37·3, it can be shown that

$$CS = e.CA, \qquad \ldots \qquad \ldots \quad (5)$$

and $$S'P - SP = 2a. \qquad \ldots \qquad \ldots \quad (6)$$

Putting $x = ae$ in (3), we have $y = \pm b\sqrt{e^2-1} = \pm \frac{b^2}{a}$.

$$\therefore \text{ the latus-rectum} = \frac{2b^2}{a}.$$

(v) The symmetry of the curve shows that there is a second focus S' on the transverse axis at the same distance from C as S; and that there is a second directrix (associated with S') parallel to the other directrix, and at the same distance from C as X. The new focus S' is at $(-ae, 0)$. The equation of the new directrix is $x = -a/e$. As in Art 37·3 (vi), it can be easily seen that there are two imaginary foci on the conjugate axis, and two corresponding directrices which are also imaginary. It can also be seen that the corresponding eccentricity is real, and is equal to $\frac{\sqrt{a^2+b^2}}{b}$.

**(vi) Putting $d = 0$ in (2), we have

$$x^2 - \frac{y^2}{e^2-1} = 0,$$

which represents a pair of real straight lines through the centre. These lines will be distinct or coincident according as e is finite, or infinitely great. Hence, *two real straight lines (distinct or coincident) may be regarded as a limiting case of the hyperbola.*

Again, proceeding as in (viii) of Art. 37·3, it can be shown that the *parabola may be regarded as a limiting case of the hyperbola.*

In the special case when $b=a$, we have $e=\sqrt{2}$. In this case, the curve is called *a rectangular hyperbola.* [5]

****44·4. On the condition that an equation of the second degree in x, y may represent a hyperbola.**

Proceeding exactly as in Arts. 37·4 and 37·4, it can be easily shown that the *necessary* and *sufficient* condition that the most general equation of the form

$$ax^2+2hxy+by^2+2gx+2fy+c=0, \quad \ldots \quad (7)$$

where a, h, b, g, f, c are arbitrary constants (none being *zero*), may represent a hyperbola is $ab<h^2$. [6]

****44·5. On the hyperbola as represented by the general equation of the second degree.**

(*i*) The *centre* of the conic represented by (7) has been found in Art. 37·5. For a hyperbola, $ab-h^2$ is *negative*, and the *centre* is at a finite distance from the origin.

(*ii*) Referred to parallel axes through the centre, the equation (7) reduces to $ax^2+2hxy+by^2=-\dfrac{\Delta}{C}$ whether the conic is an ellipse or a hyperbola, Δ and C being as in Art. 37·5. This is clear from the fact that the criterion which distinguishes the ellipse from the

[5] See foot-note 4 of Ch. V. Similar remarks apply to the results of this article.

[6] As in the case of the ellipse, *five* independent conditions (or as many of them as will give rise to five independent equations between the constants) are *necessary* and *sufficient* to determine a hyperbola.

hyperbola was not necessary in obtaining the above result (see Art. 37·6).

(*iii*) In Art. 37·61 (*i*), it has been found that the squares of the lengths of the semi-axes of the ellipse $ax^2+2hxy+by^2=1$, $(ab>h^2)$, are the roots of r^2 in

$$\left(a-\frac{1}{r^2}\right)\left(b-\frac{1}{r^2}\right)-h^2=0. \quad \ldots \quad (8)$$

It is easy to see that the same method is sufficient for us in discussing this problem for the hyperbola

$$ax^2+2hxy+by^2=1,\ (ab<h^2). \quad \ldots \quad \ldots \quad (9)$$

In this case, the product of the roots of r^2 in (8) $=1/(ab-h^2)=$ a *negative* quantity. Hence, the roots must be of different signs.[7]

(*iv*) The position of the axes of the hyperbola (9) can also be found as in Art. 37·61 (ii).

The transverse and conjugate axes correspond respectively to positive and negative values of r^2 in (8). Their equations are

$$\left(a-\frac{1}{\alpha^2}\right)x+hy=0 \quad \text{and} \quad \left(a+\frac{1}{\beta^2}\right)x+hy=0$$

respectively, α^2 and $-\beta^2$ being the roots of (8) in r^2.[8–9]

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[7] The method of foot-note 8 of Ch. VI can also be used in finding the lengths of the axes of hyperbola. Putting $-\beta^2$ for β^2, we see that the semi-axes are given by

$$\frac{1}{\alpha^2}-\frac{1}{\beta^2}=a+b \text{ and } \frac{1}{\alpha^2\beta^2}=h^2-ab.$$

[8] Note that α is not necessarily greater than β here.

[9] As in foot-note 9 of Ch. VI, the equation of the axes is given by

$$\frac{x^2-y^2}{a-b}=\frac{xy}{h}.$$

(*v*) In case of the hyperbola, we have $e^2 = \dfrac{\beta^2 + \alpha^2}{\alpha^2}$.

$$\therefore \qquad \frac{e^2}{2 - e^2} = \frac{\alpha^2 + \beta^2}{\alpha^2 - \beta^2}.$$

$$\therefore \qquad \frac{e^2}{2 - e^2} = + \sqrt{\frac{(a-b)^2 + 4h^2}{(a+b)^2}},$$

which gives e^2.

(*vi*) The foci on the transverse axis are obtained by writing $-\beta^2$ for β^2 in Art. 37·63. The foci on the conjugate axis are *imaginary*.

*44·6. The position in oblique co-ordinates.

As in Art. 37·7, it can be easily seen that $ab < h^2$ is the condition that the general equation of the second degree in x, y, may represents a hyperbola even when the axes are *oblique*.

45·7. Parametric equations of the hyperbola.

As in Art. 37·8, it can be easily seen that

$$\left.\begin{aligned} x &= a \sec\theta, \\ \text{and } y &= b \tan\theta \end{aligned}\right\} \qquad \ldots \qquad \ldots \qquad (10)$$

form a set of parametric equations of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Examples.

1. Write down the equation of the hyperbola whose focus is at the point (2, 3), whose directrix is the line $3x - 2y + 1 = 0$, and whose eccentricity $= \sqrt{3}$.

**2. Find the co-ordinates of the centre of the hyperbola $x^2 + 4xy + 3y^2 + 7x - 2y = 5$.

**3. Find the lengths and position of the axes of a hyperbola in the following cases :—

(i) $x^2+4xy+3y^2+7x-2y=5.$

(ii) $2x^2+8xy-3y^2-4x+5y=1.$

**4. Find the latus-rectum, the eccentricity and the foci of the hyperbola of Ex. 2.

[*Hints.* Reduce it to ‖ axes through C, and then use the results of Art. 44·5. (Cf. Ex. 8 of Art. 37).]

**5. Trace the curve $x^2+4xy+3y^2+2x-2y=5$.

[*Hints.* Since $ab<h^2$ in this case, the curve is a hyperbola.

The centre $C\equiv(x_1, y_1)$ is given by

$$x_1+2y_1+1=0 \text{ and } 2x_1+3y_1-1=0.$$

$\therefore$ $x_1=5$ and $y_1=-3$.

$\therefore$ The equation of the curve referred to parallel axes through the centre is

$$x^2+4xy+3y^2=-3.$$

By Art. 37·61 (i), the squares of the lengths of the semi-axes are given by

$$\left(\frac{1}{3}+\frac{1}{r^2}\right)\left(1+\frac{1}{r^2}\right)=\frac{4}{9}.$$

$\therefore$ $r^2=6+\sqrt{45}$, or $6-\sqrt{45}$.

Again, by Art. 37·61 (*ii*), the directions of the transverse and conjugate axes are given by

$$(9+\sqrt{45})x+2(6+\sqrt{45})y=0 \;\&\; (9-\sqrt{45})x+2(6-\sqrt{45})y=0.$$

Hence, the points A, A', B, B' can be found (Fig. 41). Now, find the points of intersection of the curve with the original axes (using the original equation in this case); and plot a few more points with the help of the same equation. Then, use your knowledge of the nature of the hyperbola in drawing the curve. (Art. 47·21 (*iii*) will throw additional light in this connection. Cf. Ex 8 of Art. 37 & foot-note 64 of Ch. I).]

**6. Trace the hyperbolas of Ex. 3.

7. Show that $x = a \cosh t$, $y = b \sinh t$ is a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ for all values of t.

8. Show that the straight lines $\frac{x}{a} + \frac{y}{b} = \lambda$ and $\frac{x}{a} - \frac{y}{b} = \frac{1}{\lambda}$ meet on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ for all values of λ.

45.

ON THE EQUATIONS OF THE ELLIPSE AND THE HYPERBOLA.

The standard forms of the equations of the ellipse and the hyperbola are

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ respectively. The forms of these equations show that a mere change in the sign of b^2 is sufficient for us in writing down several results for the hyperbola from the corresponding results of the ellipse.[10] Thus, we have the following results for the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$:—

(i) The equation of the tangent at (x_1, y_1) is

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[10] We have already got some hints on this point from the manner in which the results of Art. 44·5 have been obtained.

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1. \text{ [cf. Art. 38 (i).]}$$

(ii) Tangents at the extremities of any chord through the centre of a hyperbola are parallel to one another. [*cf.* Art. 38·1.].

(iii) The condition that the line $y = mx + c$ may touch the hyperbola is

$$c^2 = a^2 m^2 - b^2. \text{ [cf. Art. 38 (iii).]}$$

(iv) Two tangents (real or imaginary) can be drawn to a hyperbola from a given point outside it. [*cf* foot-note 20 of Ch. VI.]

(v) The equation of the pair of tangents from (x_1, y_1) is $SS_1 = T^2$,

$$\text{or } \left(\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1\right)\left(\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1\right) = \left(\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1\right)^2.$$

[*cf.* Art. 38 (iv).]

(vi) The equation of the chord having (x_1, y_1) as its middle point is $T = S_1$,

$$\text{or } \frac{xx_1}{a^2} - \frac{yy_1}{b^2} = \frac{x^2}{a^2} - \frac{y^2}{b^2}. \text{ [cf. Art. 39 (ii).]}$$

(vii) The equation of the chord of contact of tangents from (x_1, y_1) is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1. \text{ [cf. Art. 39 (iii).]}$$

(viii) The polar of the point (x_1, y_1) is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1. \text{ [cf. Art. 41 (i).]}$$

(ix) The pole of the line $y = mx + c$ is

$$\left(-\frac{a^2 m}{c}, -\frac{b^2}{c}\right). \text{ [cf. Art. 41 (ii).]}$$

[This does not exhaust the list of results that could have been written down in this way.][11]

Examples.

1. Find the condition that the line $x\cos\alpha + y\sin\alpha - p = 0$ may touch the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.

2. Find the equation of the normal at $P \equiv (a\sec\phi, b\tan\phi)$ on the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.

[*Solution.* The tangent at P is $\dfrac{x}{a}\sec\phi - \dfrac{y}{b}\tan\phi = 1$.

Hence, the normal is $y - b\tan\phi = -\dfrac{a}{b}\sin\phi\,(x - a\sec\phi)$,

or $ax\cos\phi + by\cot\phi = a^2 + b^2$.]

3. The condition that the line $lx + my + n = 0$ may be a normal to the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ is

$$\frac{a^2}{l^2} - \frac{b^2}{m^2} = \frac{(a^2 + b^2)^2}{n^2}. \quad [cf. \text{ Ex. 6 of Art. 40.}]$$

4. The line joining the focus of a conic to the pole of any chord through it is $\perp$ to the chord.

5. *Any chord (of a conic) through P is cut harmonically by P and its polar with respect to the conic.*

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[11] For the proof of the results (i) and (iii) to (ix), use the methods of Ch. III. For the proof of result (ii) use the method of Art. 38·1.

46.

ON DIAMETERS.

Let the equation of the hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Then, by using the methods of Art. 42, the following results can be easily obtained :—

(i) The locus of the middle points of chords of the hyperbola which are parallel to the line $y = mx$ is

$$ma^2y - b^2x = 0.$$

[It is a straight line passing through the centre of the hyperbola, and is called a *diameter* of the hyperbola.]

(ii) The condition that the lines $y = mx$ and $y = m'x$ may be conjugate diameters of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$mm' = \frac{b^2}{a^2}.$$ [12]

46·1. Some geometrical theorems.

(i) *Of two conjugate diameters of a hyperbola, one meets the curve in real points, and the other in imaginary points.*

Let the equation of the hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

The line $y = mx$ meets the hyperbola in points whose abscissæ are given by

$$x^2\left(\frac{1}{a^2} - \frac{m^2}{b^2}\right) = 1. \quad \dots \quad \dots \quad (1)$$

[12] Compare these results with the corresponding results of Art. 42.

And since the diameter conjugate to $y=mx$ is $ma^2y-b^2x=0$, it meets the hyperbola in points whose abscissæ are given by

$$\frac{x^2}{a^2}\left(1-\frac{b^2}{a^2m^2}\right)=1. \qquad \dots \qquad \dots \quad (2)$$

Now, if the roots of (1) in x are *real*, then we must have $b^2>a^2m^2$.

But, when $b^2>a^2m^2$, the roots of (2) in x are evidently *imaginary*. Hence the theorem.

(ii) *The tangent at the extremity of any diameter is parallel to the conjugate diameter.*

Let $P\equiv(a\sec\phi,\ b\tan\phi)$ be one extremity of any diameter. Then, the equation of the diameter conjugate to CP is $a\sin\phi.y-bx=0$; and that of the tangent at P is

$$\frac{x}{a}\sec\phi-\frac{y}{b}\tan\phi=1.$$

The m's of both the lines being $=b/a\sin\phi$, the lines are evidently parallel to each other.

(iii) *The chords joining any point on a hyperbola to the extremities of any diameter are parallel to a pair of conjugate diameters.*

Let $P\equiv(a\sec\phi,\ b\tan\phi)$ by any point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$; and let $Q\equiv(a\sec\alpha,\ b\tan\alpha)$ and $Q'\equiv(-a\sec\alpha,\ -b\tan\alpha)$ be the extremities of the diameter QCQ'. Then, the 'm' of the line PQ is $\frac{b(\tan\alpha-\tan\phi)}{a(\sec\alpha-\sec\phi)}$;

and that of the line PQ' is $\frac{b(\tan\alpha+\tan\phi)}{a(\sec\alpha+\sec\phi)}$.

The product of the m's of these lines is $= \frac{b^2}{a^2}$. Hence, PQ and PQ' are parallel to a pair of conjugate diameters.

16·2. If P and D be the extremities of a pair of conjugate semi-diameters of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then $CP^2 + CD^2 = a^2 - b^2$.

Let $P \equiv (a \sec\phi, b \tan\phi)$. Then, the equation of CD is $y = \frac{bx}{a \sin\phi}$. Hence, $D \equiv (ai \tan\phi, bi \sec\phi)$, say. ($i = \sqrt{-1}$).

$$\therefore \qquad CP^2 = a^2 \sec^2\phi + b^2 \tan^2\phi.$$

$$\text{and} \qquad CD^2 = -(a^2 \tan^2\phi + b^2 \sec^2\phi).$$

$$\therefore \qquad CP^2 + CD^2 = a^2 - b^2.$$

Examples.

**1. Find the equation of a diameter of the central conic $ax^2 + 2hxy + by^2 = 1$.

[*Solution.* The equation of a chord having (x_1, y_1) as its middle point is $T = S_1$,

or $axx_1 + h(xy_1 + x_1y) + byy_1 = ax_1^2 + 2hx_1y_1 + by_1^2$.

If it is parallel to the line $y = mx$, we must have

$$m = -(ax_1 + hy_1)/(hx_1 + by_1).$$

Hence, the equation of the diameter bisecting all chords (*i.e.*, the locus of the middle points of chords) parallel to the line $y = mx$ is

$$x(a + mh) + y(h + mb) = 0. \qquad \ldots \qquad \ldots \qquad (3)]$$

**2. Find the condition under which the lines $y = mx$ and $y = m'x$ may be conjugate diameters of the central conic $ax^2 + 2hxy + by^2 = 1$.

[*Solution.* The diameter bisecting all chords parallel to $y=mx$ is given by (3). If it be the same as $y=m'x$, we must have

$$m' = -(a+mh)/(h+mb).$$

$$\therefore \quad a+(m+m')h+mm'b=0,$$

which is the condition of conjugacy.]

**3. Show that the condition that the lines $Ax^2 + 2Hxy + By^2 = 0$ may be conjugate diameters of the conic $ax^2+2hxy+by^2=1$ is $aB+bA=2hH$.

**4. Find the condition under which the lines $y=mx$ and $y=m'x$ may be conjugate diameters of the conic represented by the general equation of the second degree.

5. Show that the equation of the hyperbola with reference to a pair of conjugate diameters as axes is of the form $$\frac{x_1^2}{a_1^2}-\frac{y_1^2}{b_1^2}=1.$$

[*Hints.* Proceed as in Art. 42·5.]

6. Show that conjugate diameters of a hyperbola lie on the same side of the conjugate axis.

47.

THE ASYMPTOTES.

47·1. Definition.

An *asymptote* to a curve is a straight line which meets the curve at two points at infinity, but which does not lie wholly at infinity. [An *asymptote* may be regarded as a tangent at infinity.]

47·2. To find the asymptotes of the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1. \quad \ldots \quad \ldots \quad (1)$$

Let the equation of an asymptotes be $y = mx + c$.

Putting $y = mx + c$ in (1), we have

$$\frac{x^2}{a^2} - \frac{(mx+c)^2}{b^2} = 1.$$

$$\therefore \quad x^2\left(\frac{1}{a^2} - \frac{m^2}{b^2}\right) - \frac{2mcx}{b^2} - \frac{c^2}{b^2} - 1 = 0. \qquad \ldots (2)$$

Now, the line $y = mx + c$ will meet the hyperbola in two points at infinity, if the roots of (2) in x be both *infinite*. In this case, we must have

$$\frac{1}{a^2} - \frac{m^2}{b^2} = 0 \text{ and } mc = 0.$$

$$\therefore \quad m = \pm\frac{b}{a} \text{ and } \quad c = 0.$$

Hence, the asymptotes are given by

$$y = \pm\frac{b}{a}x, \qquad \ldots \qquad \ldots (3)$$

$$\text{or} \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 0. \qquad \ldots \qquad \ldots (4)$$

47·21. On some points connected with the asymptotes.

(*i*) The equation of the pair of asymptotes of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ as given by (4) differs from that of the hyperbola itself by a constant. And as the terms in x, y are similarly affected in both by any transformation of Cartesian co-ordinates, it is clear that *the above result will be true whatever may be the form of the equation of the curve.*

(*ii*) From (3), it is clear that the asymptotes are equally inclined to either axis of the curve. So, the angle between the asymptotes $=2\tan^{-1}\frac{b}{a}$. For a rectangular hyperbola, we have $a=b$. Hence, it is clear from the above that the asymptotes of a rectangular hyperbola are at right angles to each other.

(*iii*) By constructing the rectangle formed by drawing lines though B and B' parallel to the transverse axis, and through A and A' parallel to the conjugate axis, it can be easily seen that *the diagonals of the rectangle so formed lie along the asymptotes of the hyperbola.*[12]

(*iv*) Any straight line parallel to an asymptote will meet the curve at infinity in *only one point*. For, the equation of such a line being $y=\frac{b}{a}x+c$, the abscissæ of the points of intersection of the line and the hyperbola are given by

$$\frac{2cx}{ab}+\frac{c^2}{b^2}+1=0.$$

And as one root of this equation is infinite for all values of c (and irrespective of the sign of b/a), the truth of the above theorem is established.[13-14]

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[12] Hence, the asymptotes of a hyperbola can be *mechanically* constructed very easily, if the lengths and position of the axes of the curve are given.

[13] It can be easily seen that the asymptotes of an ellipse are *imaginary*. Hence, these lines have no geometrical significance for the ellipse.

[14] If a line cuts the parabola in two points at infinity, the line is altogether at infinity. This can be easily verified by taking $y^2=4ax$ as the equation of the parabola and $lx+my=1$ as that of the line.

****47·22. To find the asymptotes of the central conic $ax^2+2hxy+by^2+2gx+2fy+c=0$, $(h^2 \neq ab)$.**

From Art. 47·21 (i), its clear that the equation of the pair of asymptotes differs from that of the central conic by a constant. Hence, the equation of the asymptotes may be written as

$$ax^2+2hxy+by^2+2gx+2fy+c=\lambda, \qquad \ldots (4)$$

where λ is to be chosen so that (4) may represent a pair of straight lines.[15] The condition for this is

$$\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c-\lambda \end{vmatrix} = 0.$$

$$\therefore \quad \lambda = \frac{\Delta}{C}, \qquad \text{where } \Delta \text{ and } C \text{ are as in Art. } 37·6.$$

Hence, the equation of the pair of asymptotes is

$$ax^2+2hxy+by^2+2gx+2fy+c=\frac{\Delta}{C}. \qquad \ldots (5)$$

****47·3. To find the equation of the hyperbola with reference to its asymptotes as axes.**

The asymptotes pass through the centre of the hyperbola. Hence, if the asymptotes be taken as the axes of co-ordinates, the centre of the conic must be at the origin. Hence, by Art. 44·5 (ii), the equation of the hyperbola must be of the from

$$ax^2+2hxy+by^2=1, \quad (ab<h^2).$$

Now, putting $x=0$, we have $y^2=\frac{1}{b}$.

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15 By definition, an asymptote is a straight line (of course, connected with a curve as in Art. 47·1).

And since the axis of y is an asymptote, both the roots of this equation must be infinite. Hence, $b = 0$. Similarly, it can be seen that a must be zero in order that the axis of x may be an asymptote.

Hence, the equation of the hyperbola with reference to its asymptotes as axes of co-ordinates is of the form

$$xy = k^2. \quad \ldots \quad \ldots \quad (6)$$

*47·31. Notes on the above.

(*i*) Since the axes of the hyperbola bisect the angles between the asymptotes, the co-ordinates of the vertices (with reference to the asymptotes as axes) A and A' are obtained by combining $x = y$ with (6). Now, let

$$A \equiv (k, k) \text{ and } A' \equiv (-k, -k).$$

$\therefore \quad CA^2 = 2k^2 + 2k^2 \cos 2\theta$, if 2θ be that angle between the asymptotes which contain the transverse axis.

But $\quad CA^2 = a^2$, by Art. 44·3 (*ii*).

$\therefore \quad 4k^2 \cos^2\theta = a^2. \qquad$ Also $\tan \theta = \dfrac{b}{a}$, by (3).

$$\therefore \quad k^2 = \frac{a^2 + b^2}{4}.$$

(*ii*) From (6), is is clear that $P \equiv \left(kt, \dfrac{k}{t}\right)$ is a point on the hyperbola $xy = k^2$ for all values of t.

(*iii*) It can be easily seen that the equation of the chord joining P and $Q \equiv \left(kt_1, \dfrac{k}{t_1}\right)$ is $x + ytt_1 = k(t + t_1)$.

Hence, the equation of the tangent at 't' on the hyperbola $xy = k^2$ is $\qquad x + yt^2 = 2kt.$

Examples.

1. Find the asymptotes of a hyperbola in the following cases :—

(i) $x^2+4xy+y^2=1$. (ii) $2x^2+3xy-2y^2-5x+y-3=0$.

(iii) $(ax+by+c(a'x+b'x+c')=\lambda^2$.

2 Find the equation of the hyperbola whose asymptotes are $x+y+1=0$ and $3x-4y+2=0$ and which passes through the point $(3, -2)$.

*3. *The tangent at any point of a hyperbola cuts off a triangle of constant area from the asymptotes.*

[*Solution.* Let the asymptotes be taken as the axes of co-ordinates. Then, the equation of the hyperbola is of the form $xy=k^2$. Now, the tangent at any point 't' is $x+yt^2=2kt$. It meets the axes at $(2kt, 0)$ and $(0, 2k/t)$. Hence, by Art. 6·4, it can be easily seen that the area of the $\triangle =2k^2 \sin \omega =$ a constant.]

4. Show that the portion of the tangent to a hyperbola intercepted between the asymptotes is bisected at the point of contact, and is equal to the diameter conjugate to that to the point of contact.

5. P and Q are two fixed points on a given hyperbola and D any point on it. PD and QD are produced to meet an asymptote in R and R' respectively ; show that RR' is constant.

6. Show that the asymptotes of a central conic may be regarded as the pair of tangents from its centre to the conic.

7. Find the equation of the axes of the central conic $ax^2+2hxy+by^2=1$ by using the fact that the axes are the bisectors of the angles between the asymptotes.

48.

THE HYPERBOLA AND ITS CONJUGATE.

Let the equation of a hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Then, the equation of the hyperbola whose transverse and conjugate axes are the same as the conjugate and transverse axes respectively of the given hyperbola is

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1, \quad \text{or} \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = -1.$$

This hyperbola is said to be *conjugate* to the original hyperbola.

[If AA', BB' be the transverse and conjugate axes of the given hyperbola (fig. 41), then BB' and AA' are the transverse and conjugate axes of the conjugate hyperbola.]

48·01. Notes on the above.

(i) From the above, it is clear that a hyperbola and its conjugate have the same asymptotes.

(ii) Now, comparing the equation of the hyperbola with the equations of its asymptotes and the conjugate hyperbola, we see that the equation of the asymptotes differs from that of the hyperbola by the same constant as the equation of the conjugate hyperbola differs from that of the asymptotes. And as any transformation of Cartesian co-ordinates will alter the terms in x and y in exactly the same manner, it is clear that the above relationship between these equations remains unaltered by any transformation of Cartesian axes.

(iii) The straight lines $y = mx$ and $y = m'x$ are conjugate diameters of the first hyperbola, if $mm' = \frac{b^2}{a^2}$. But this is also the condition that the lines may be conjugate to the second hyperbola. Hence, *a pair of conjugate diameters of a hyperbola is also a conjugate pair for the conjugate hyperbola.*

(iv) Again, *if the diameter $y = mx$ cuts the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ in real points, then the conjugate diameter $ma^2y - b^2x = 0$ cuts the conjugate hyperbola $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ in real points.* For, the abscissæ of the points of intersection in the first case is given by

$$\frac{x^2}{a^2}\left(1 - \frac{a^2m^2}{b^2}\right) = 1; \quad \ldots \quad \ldots \quad (1)$$

and those in the second case by

$$\frac{x^2}{a^2}\left(\frac{b^2}{a^2m^2} - 1\right) = 1. \quad \ldots \quad \ldots \quad (2)$$

And since $b^2 > a^2m^2$ is the condition for real roots in either case, it is clear that (2) is satisfied, if (1) is satisfied; and vice versa. Hence the theorem.

48·1. If a pair of conjugate semi-diameters of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meet it at P_1 and its conjugate at D_1, then $CP_1^2 - CD_1^2 = a^2 - b^2$.

Let $P_1 \equiv (a \sec\theta, b \tan\theta)$ be any point on the hyperbola. Then, the 'm' of CP_1 is $\frac{b \sin\theta}{a}$; and by Art. 46 (ii), the equation of the diameter conjugate to CP_1 is

$$y = \frac{bx}{a \sin \theta}.$$

Now, one of the points where it meets the conjugate hyperbola $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ is

$$D_1 \equiv (a \tan \theta, b \sec \theta), \text{ say.}$$

$\therefore \qquad CD_1{}^2 = a^2 \tan^2 \theta + b^2 \sec^2 \theta.$

also $\qquad CP_1{}^2 = a^2 \sec^2 \theta + b^2 \tan^2 \theta.$

$\therefore \; CP_1{}^2 - CD_1{}^2 = a^2 - b^2.$

Examples.

1. If e and e' be the eccentricities of a hyperbola and its conjugate, then $\frac{1}{e^2} + \frac{1}{e'^2} = 1$.

2. Find the equation of the hyperbola conjugate to a hyperbola in the following cases :—

(i) $x^2 - y^2 = a^2$. **(iii) $x^2 + 4xy + 3y^2 = 6$.

(ii) $x(x+y) = 1$. **(iv) $2x^2 + 5xy - 3y^2 + 5x - 4y + 7 = 0$.

3. If a pair of conjugate diameters of a hyperbola cut it and its conjugate at P, P_1, and D, D_1, then the tangent at these points form a parallelogram of constant area.

4. *Any chord of a hyperbola which touches its conjugate is bisected at the point of contact.*

49.

THE RECTANGULAR HYPERBOLA.

For a rectangular hyprebola, $a = b$ (that is, $e = \sqrt{2}$). Hence, the equation of a rectangular hyperbola may be written as

$$x^2 - y^2 = a^2, \qquad \ldots \qquad \ldots \quad (1)$$

the axes being rectangular.

The asymptotes are given by $x^2 - y^2 = 0$, and are evidently *perpendicular* to each other.

The equation of the rectangular hyperbola with reference to its asymptotes as axes (rectangular) is

$$xy = \frac{a^2}{2} = c^2, \text{ say.} \qquad \ldots \qquad \ldots \quad (2)$$

The point $(ct, \frac{c}{t})$ is a point on the rectangular hyperbola for all values of t. [Cf. Art. 47·31 (i).][16]

****49·1. To Find the condition that the conic $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, (ab < h^2)$, may be a rectangular hyperbola.**

In a rectangular hyperbola, the asymptotes are at right angles to each other. Now, since the equation of the asymptotes differs from that of the conic by a constant, it is clear from (18) of Art. 9·41 that the asymptotes are parallel to the lines $ax^2 + 2hxy + by^2 = 0$. And the condition of perpendicularity of these lines is $a + b = 0$. Hence, the required condition is

$$a + b = 0.$$

49·2. Some geometrical theorems.

In a rectangular hyperbola, the line joining the centre to any point on the curve is a mean proportional between the focal distances of the point.

Let the equation of the rectangular hyperbola be $x^2 - y^2 = a^2$; and let $P \equiv (a \sec\phi, a \tan\phi)$ be any point on it.

---

[16] Note the difference in the values of c and k.

Now, the foci are $S \equiv (a\sqrt{2}, 0)$ and $S' \equiv (-a\sqrt{2}, 0)$.

$\therefore$ $SP = a(\sqrt{2}\sec\phi - 1)$ and $S'P = a(\sqrt{2}\sec\phi + 1)$.

Also $CP^2 = a^2(2\sec^2\phi - 1)$.

$\therefore$ $SP.S'P = a^2(2\sec^2\phi - 1) = CP^2$.

Hence the theorem.

(ii) *If a rectangular hyperbola circumscribes a triangle, it passes through the orthocentre of the triangle.*

Let the equation of the hyperbola be $xy = c^2$.

Also let the angular points of the triangle be

$$P \equiv \left(ct_1, \frac{c}{t_1}\right),\ Q \equiv \left(ct_2, \frac{c}{t_2}\right) \text{ and } R \equiv \left(ct_3, \frac{c}{t_3}\right).$$

Then, the equation of PQ is $x + yt_1t_2 = c(t_1 + t_2)$.

So, the equation of the line through R perpendicular to PQ is $y + ct_1t_2t_3 = t_1t_2\left(x + \dfrac{c}{t_1t_2t_3}\right)$.

Similarly, the equation of the straight line through Q perpendicular to PR is

$$y + ct_1t_2t_3 = t_1t_3\left(x + \frac{c}{t_1t_2t_3}\right).$$

These two lines meet at the orthocentre D, where

$$D \equiv \left[-\frac{c}{t_1t_2t_3}, -ct_1t_2t_3\right].$$

Now, D is clearly a point on $xy = c^2$. Hence the theorem.

Examples.

1. In a rectangular hyperbola, the angle between the tangents at the extremities of a chord is supplementary to the angle subtended by the chord at the centre of the curve.

2. The normal at P on a rectangular hyperbola meets the axes in G and g_1 show that $PG = Pg_1 = PC$.

3. If a circle cuts the rectangular hyperbola $xy = c^2$ in four points, the product of the abscissæ of the four points is constant.

4. The normal to the rectangular hyperbola $xy = c^2$ at the point 't_1' meets the curve again at the point 't' show that $t = -\dfrac{1}{t_1^3}$.

5. PQR is a triangle right-angled at Q and inscribed in a rectangular hyperbola; show that the tangent at Q is perpendicular to PR.

6. It two conjugate diameters of a rectangular hyperbola meet it and its conjugate at P_1, D_1, then $CP_1 = CD_1$.

50.

ON SOME LOCUS PROBLEMS CONNECTED WITH THE HYPERBOLA.

As we have had long discussions of locus problems in the previous chapters, and have nothing useful to add, we conclude this section by setting some problems for the student. [It should be a very instructive exercise for the student to consider them as additional examples under Art. 23, 35 or 43, and solve them accordingly. Of course, our knowledge of the hyperbola would be useful to us in this connection.

Examples.

1. Find the locus of the point of intersection of tangents to a hyperbola which cut at right angles.

2. Find the locus of the point of intersection of two normals to a hyperbola which cut at right angles.

3 Show that the locus of the poles of normal chords of the rectangular hyperbola $xy = c^2$ is given by

$$(x^2 - y^2)^2 + 4c^2 xy = 0.$$

4. Find the locus of the foot of the perpendicular from the centre of a rectangular hyperbola to any tangent to it.

5. Show that the locus of the middle points of chords of the rectangular hyperbola $xy = c^2$ which are of constant length $2l$ is

$$(x^2 + y^2)(xy - c^2) - l^2 xy = 0.$$

6. Find the locus of the centre of the rectangular hyperbola which circumscribes a given triangle.

7. Find the locus of the poles of chords of a hyperbola which subtend a right angle at its centre.

CHAPTER VIII.

MISCELLANEOUS TOPICS CONNECTED WITH CONIC SECTIONS IN GENERAL. [1]

51.

ON POLAR EQUATIONS OF CONICS.

By Arts. 3·2 and 3·3, we can find the polar equation of a conic when its Cartesian equation is given. [2] But when the *conic* is referred to its *focus* as the *pole*, its polar equation assumes a very simple form (as will be seen presently).

[We have already made use of our knowledge of polar co-ordinates in solving certain problems. [3] In this section, we shall illustrate the usefulness of the polar equation of a conic as stated above.][4–5]

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[1] In this chapter, we shall discuss briefly some of the remaining topics connected with conics in general (see Ch. IV).

[2] Note that the polar equations of the straight line and the circle have been obtained in this way in Arts. 9·5 and 17·5 respectively.

[3] See Arts. 35·3, 43·3, etc.

[4] It is not that problems solvable with the help of the polar equation as stated above cannot be solved with the help of Cartesian co-ordinates. As a matter of fact, the discussion of the question of solving a problem with the help of polar co-ordinates has been very brief in order that the attention of the students may be drawn more towards the methods of Cartesian co-ordinates already discussed than towards those of the polar co-ordinates.

[5] Note that the polar equation of the conic as stated above can be used with advantage in solving many problems on the focal properties of conic sections.

51·1. To find the polar equation of a conic, its focus S being the pole.

Let P be any point on the conic, and let its polar co-ordinates with reference to S as the pole and XS as the initial line be r and θ respectively (fig. 34).

Now, by definition,

$$r = SP = e.PM = e.XN.$$

$$= e(XS + SN).$$

$$= e\left(\frac{l}{e} + r \cos\theta\right) = l + e\,r \cos\theta,$$

where l = semi-latus-rectum.

$$\therefore \quad \frac{l}{r} = 1 - e \cos\theta. \quad \ldots \quad (1)$$

This is a relation between the polar co-ordinates of any point on the curve, and is *free* from any other unknown quantity. Hence, (1) is the polar equation of the conic as desired.

51·11. Notes on the above.

(i) The curve represented by (1) can be easily traced by the fundamental method of finding the values of r corresponding to suitably assigned values of θ, and noting the changes in the values of r as θ increases from $0°$ to $90°$, $90°$ to 180, $180°$ to $270°$ and $270°$ to $360°$ respectively. [It need hardly be added that the curve will be an ellipse, parabola, or hyperbola according as $e <, =,$ or > 1.]

(ii) If $e = 1$, (1) becomes $\frac{l}{r} = 1 - \cos\theta = 2 \sin^2 \frac{\theta}{2}$ ...

In this case, r is *positive* for all values of θ. [See Art. 2·3].

(iii) If α be the inclination of the initial line to the line XS, then it can be easily seen from fig. 34 that the equation of the conic, in this case, is

$$\frac{l}{r}=1-e\cos(\theta-\alpha).$$

(iv) The polar equation of the chord joining the points on the conic (1) whose vectorial angles are $\alpha-\beta$ and $\alpha+\beta$ is

$$\frac{l}{r}=\sec\beta\cos(\theta-\alpha)-e\cos\theta. \quad \ldots \quad (2)$$

Hence, the polar equation of the tangent at the point on the conic whose vectorial angle is α is obtained by putting $\beta=0$ in (2).[7]

51·2. Some geometrical theorems.

(*i*) *The semi-latus rectum of a conic is a harmonic mean between the segments of any focal chord.*

Let the equation of the conic be $\frac{l}{r}=1-e\cos\theta$.

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[6] This truth of this result can be seen as follows:—Changing (2) to Cartesian co-ordinates, it can be easily seen that it represents a straight line. Again, putting $\theta=\alpha-\beta$ in (2), we have $l/r=1-e\cos(\alpha-\beta)$. Hence, the point on the conic whose vectorial angle is $\alpha-\beta$ lies on the line (2). Similarly, it can be seen that the same is true of the point on (1) whose vectorial angle is $\alpha+\beta$.

[Note that the method used is the same as that of Art. 19·1.]

[7] As a mere change of co-ordinates cannot change the nature of the geometrical relationship between a curve and a line (say, a tangent at some point on it), it is clear that $a/r=m\sin\theta-m^2\cos\theta$ touches the curve $4a/r=\sec\theta-\cos\theta$, since $y=mx+a/m$ touches the curve $y^2=4ax$; and so on.

Also let PSP' be a focal chord, and θ be the vectorial angle of P. Then, $\theta+\pi$ is the vectorial angle of P'.

$$\therefore \quad \frac{l}{SP}=1-e\cos\theta \text{ and } \frac{l}{SP'}=1+e\cos\theta.$$

$$\therefore \quad \frac{l}{SP}+\frac{l}{SP'}=2, \text{ or } \frac{1}{SP}+\frac{1}{SP'}=\frac{2}{l},$$

which proves the theorem.

(*ii*) *If a circle passing through the focus of a conic cuts it in four points then the sum of the reciprocals of the distances of these points from the focus is a constant.*

Let the equation of the conic be $\frac{l}{r}=1-e\cos\theta$; and let α be the angle which the diameter of the circle passing through S makes with XS. Then, the equation of the circle is

$$r=2c\cos(\theta-\alpha),$$

where c is the radius of the circle.

Eliminating θ from these equations, we have

$$e^2r^4-4ce\,r^3\cos\alpha+r^2(4c^2+4cel\cos\alpha-4e^2c^2\sin^2\alpha)$$
$$-8c^2lr+4c^2l^2=0. \quad \ldots \quad \ldots \quad (3)$$

$$\therefore \quad r_1r_2r_3r_4=\frac{4c^2l^2}{e^2} \text{ and } \Sigma r_2r_3r_4=\frac{8c^2l}{e^2},$$

r_1, r_2, r_3 and r_4 being the roots of (3) in r.

$$\therefore \quad \frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4}=\frac{2}{l}, \text{ by division.}$$

Hence the theorem.

Examples.

1. *Show that the sum of the reciprocals of two perpendicular focal chords of a conic is constant.*

2. If PSP' and QSQ' be two perpendicular focal chords of a conic, then

$$\frac{1}{PS.SP'}+\frac{1}{QS.SQ'}=\text{a constant.}$$

3. If a circle of constant radius passing through the focus of a conic cuts it in four points, then the product of the focal distances of these points is constant.

4. Find the locus of the middle points of focal chords of a conic.

5. A chord of a conic subtends a constant angle at its focus; find the locus of the point of intersection of the tangents at the extremities of the chord.

**52.

ON CONICS THROUGH THE INTERSECTIONS OF TWO CONICS.

Let the equations of the two conics be

$$S\equiv ax^2+2hxy+by^2+2fy+2gx+c=0 \quad \ldots \quad (1)$$

and $S'\equiv a'x^2+2h'xy+b'y^2+2f'y+2g'x+c'=0$. ... (2)

Then, $S-\lambda S'=0$ (where λ is any constant) represents a conic passing through the four points of intersection (real or imaginary) of the two conics $S=0$ and $S'=0$ (see Art. 8·11). [8–9]

[8] $S+\lambda S'=0$ represents a conic, since it is an equation of the second degree in x, y. It passes through the intersections of (1) and (2), since it is satisfied (for all values of λ) whenever $S=0$ and

From what we have said above, it is clear that the above result will be true, if either or both the expressions S and S' are resolvable into factors.[10] Hence, :

(*i*) If $S' \equiv uu'$, where $u \equiv lx+my+n$ and $u' \equiv l'x+m'y+n'$, then $S-\lambda uu'=0$ represents a conic passing through the four points (real or imaginary) where the conic $S=0$ is cut by the straight lines $u=0$ and $u'=0$.

(*ii*) Also if $S \equiv vv'$, where $v \equiv px+qy+r$ and $v' \equiv p'x+q'y+r'$. Then, $vv'-\lambda uu'=0$ represents a conic passing through the four vertices of the quadrilateral formed by the lines $v=0$, $u=0$, $v'=0$ and $u'=0$ taken in order.[11]

52·1. Some geometrical theorems.

(*i*) *All conics passing through the points of intersection of two rectangular hyperbolas are rectangular hyperbolas.*

If the two conics $S=0$ and $S'=0$ (where S and S' are as in Art. 52) be rectangular hyperbolas, then we must have

$$\left.\begin{aligned} a+b=0 \\ \text{and } a'+b'=0. \end{aligned}\right\} \quad \ldots \quad \ldots \quad (1)$$

Now, the equation of any conic which passes through the points of intersection of $S=0$ and $S'=0$ is $S-\lambda S'=0$.

$S'=0$ simultaneously; and there are *four* points of intersection (real or imaginary), since both (1) and (2) are of the second degree in x, y.

[9] As λ is an arbitrary constant, we can choose it so that the conic $S+\lambda S'=0$ may satisfy another condition (which should be such that it may give rise to one, and only one, equation between the constants).

[10] As the result was obtained without making any consideration of this kind, it must be true in all such cases.

[11] Note that there is no loss of generality in denoting the sides of the quadrilateral in the order mentioned above.

And this will be a rectangular hyperbola, if

$$(a-\lambda a')+(b-\lambda b')=0,$$

$$\text{or } a+b-\lambda(a'+b')=0.$$

But, by (1), this is true for all values of λ. Hence the theorem.

(ii) *Of the conics passing through the points of intersection of two given conics, two are generally parabolas, and three degenerate into pairs of straight lines.*

Let the equations of the two given conics be $S=0$ and $S'=0$, where S and S' are as in Art. 52. Then, the equation of any conic passing through their points of intersection is $S-\lambda S'=0$.

Now, it represents a parabola, if

$$(a-\lambda a')(b-\lambda b')=(h-\lambda h')^2.$$

This being a quadratic equation in λ, it is clear that there are, in general, two parabolas (real or imaginary) passing through the intersections of $S=0$ and $S'=0$.

Again, it represents a pair of straight lines, if

$$\begin{vmatrix} a-\lambda a', & h-\lambda h', & g-\lambda g' \\ h-\lambda h', & b-\lambda b', & f-\lambda f' \\ g-\lambda g', & f-\lambda f', & c-\lambda c' \end{vmatrix}=0.$$

This being a cubic equation in λ, it is clear that there are, in general, three pairs of straight lines passing through the intersections of $S=0$ and $S'=0$.

52·2. Notes on the equation $S=\lambda uu'$.[12]

Let the points where the lines $u=0$ and $u'=0$ cut the conic $S=0$ be denoted by P, Q and P', Q' respectively. Now,

[12] See Art. 52 (i).

(*i*) If the point Q coincides with P (and P', Q' are separate points), then the line $u = 0$ becomes a tangent to the conic $S = 0$. In this case, $S = \lambda uu'$ represents a conic touching $S = 0$ where $u = 0$ touches it (*i. e.*, at P).[13]

(*ii*) If the points P' and Q coincide with P (and Q' is a separate point), then $u = 0$ touches $S = 0$, and $u' = 0$ passes through their point of contact. In this case, $S = \lambda uu'$ is said to have '*a contact of the second order*' with $S = 0$ (or to *osculate* $S = 0$) at the point P.[14]

(*iii*) If P', Q, Q' all coincide with P, then the tangent $u = 0$ coincides with $u' = 0$. In this case, $S = \lambda u^2$ is said to have '*a contact of the third order*' with $S = 0$ at P.

(*iv*) If Q coincides with P and Q' with P', then $u = 0$ and $u' = 0$ touches $S = 0$ at P and P' respectively. In this case, $S = \lambda uu'$ is said to have '*a double contact*' with $S = 0$ at the points P and P' (*i.e.*, where $u = 0$ and $u' = 0$ touch $S = 0$).

(*v*) If P' coincides with P, and Q' coincides with Q, then $u = 0$ coincides with $u' = 0$. In this case, $S = \lambda u^2$ is said to have '*a double contact*' with $S = 0$ at the points P and Q (*i.e.*, where $u = 0$ cuts $S = 0$).

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[13] In this case, $S = \lambda uu'$ and $S = 0$ are said to have '*a contact of of the first order*' at P.

[14] Note that the most general equation of a conic osculating $S = 0$ at the point $P \equiv (x_1, y_1)$ may be written as $S = (lx + my + n) \times (l_1 x + m_1 y - l_1 x_1 - m_1 y_1)$, where $u \equiv lx + my + n = 0$ touches $S = 0$ at P. This is clear from the fact that $u' = 0$ is now a line passing through the point P on $u = 0$.

52·3. Note on the equation $vv' = \lambda uu'$.[15]

We know that u is proportional to the length of the perpendicular drawn from any point $T \equiv (x, y)$ to the straight line $u = 0$; and so on. Using this result in connection with any point T on the conic $vv' = \lambda uu'$, we have the following theorem :—

If a conic circumscribes a quadrilateral, the product of the perpendiculars drawn from any point on it to two opposite sides of the quadrilateral bears a constant ratio to the product of the perpendiculars drawn from the same point to the other two sides.[16]

Examples.

1. Find the equation of the conic which passes through the origin, and also through the points of intersection of the conics of Art. 52.

2. The common chords a conic and a circle (taken in pairs) are equally inclined to the axes of the conic.

3. If two rectangular hyperbolas intersect in four points, then each of the points is the orthocentre of the triangle formed by the other three.

4. Use the result of foot-note 14 in finding the equation of the circle osculating a conic at any point on it in the following cases :—

(i) $y^2 = 4ax$. (ii) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

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15 See Art. 52 (ii).

16 We can obtain many other theorems with the help of the equation $vv' = \lambda uu'$ by supposing two or more of the vertices of the quadrilateral to coincide, or by imposing other suitable conditions.

[*Hints.* The equation of any such conic osculating $y^2 = 4ax$ at (x_1, y_1) is given by

$$y^2 - 4ax = \{yy_1 - 2ax - 2ax_1\}(lx + my - lx_1 - my_1).$$ [17]

Now, express the conditions under which the above equation may represent a circle, find l and m from them, and finally substitute their values in the above equation.] [18-19]

5. Solve the problem of Ex. 4 (i) by using the Differential Calculus method.

[*Solution.* The normal at any point 't' on the parabola $y^2 = 4ax$ is given by $y + tx = 2at + at^3$. ... ... (2)

Differentiating this with respect to t, we have

$$x = 2a + 3at^2 \quad \ldots \quad \ldots \quad (3)$$

at the point where (2) meets the consecutive normal.

Solving (2) and (3) for x and y, we see that the centre of curvature at 't' is $(2a + 3at^2, -2at^3)$.

Again, (radius of curvature)^2

$$= (at^2 - 2a - 3at^2)^2 + (2at + 2at^3)^2.$$
$$= 4a^2(1 + t^2)^3 = k^2 (say).$$

Hence, the equation of the osculating circle at 't' is

$$(x - 2a - 3at^2)^2 + (y + 2at^3)^2 = k^2.]$$

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[17] The tangent at (x_1, y_1) being $yy_1 = 2a(x + x_1)$.

[18] If a circle *osculates* a conic at P, then the centre of the circle is called the *centre of curvature* at P, and the radius of the circle is called the *radius of curvature* at P (see Edwards' Differential Calculus, Ch. X).

[19] By using the methods of *Differential Calculus*, we can find the centre and the radius of curvature at any point. The equation of the osculating circle at any point can, therefore, be written down at once if we know the centre and the radius of curvature at that point.

6. Find the equation of the circle having double contact with the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the ends of a latus-rectum.

7. Find the centre, the radius and the equation of the osculating circle at the point 'θ' on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

8. Show that that the circles of curvature at the vertices of a conic have contacts of the third order with the conic.

9. Show that the product of the perpendiculars drawn from any point on a conic to two given tangents to it bears a constant ratio to the square of the perpendicular on the chord of contact.

[*Hints.* Use the equation $S=\lambda u^2$ properly.]

*10. Show that the equation of a conic referred to two tangents to it as axes of co-ordinates can be written as

$$xy=\lambda\left(\frac{x}{a}+\frac{y}{b}-1\right)^2$$

11. Find the equation of the pair of tangents to the ellipse $S\equiv\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$ from the point $P\equiv(x_1, y_1)$ outside it by considering it as a conic having double contact with $S=0$.

12. Find the equation of the director circle of the conic $S\equiv ax^2+2hxy+by^2+2gx+2fy+c=0$.

13. Use your knowledge of the result of Art. 52 (i) in solving Ex. 6 of Art. 37.

14. Show that the polar of a given point with respect to a system of conics through four fixed points passes through another fixed point.

15. Find the equation of any conic osculating the conic $ax^2 + 2hxy + by^2 = 2x$ at the origin. Hence (or otherwise), find the equation of the circle of curvature at the origin.

**16. Three points on the parabola $y^2 = 4ax$ are taken so that the centres of curvature at these points may be collinear, show that the sum of the reciprocals of their ordinates is zero.

**17. If ρ and ρ_1 be the radii of curvature at the ends of a focal chord of the parabola $y^2 = 4ax$, then

$$\rho^{-\frac{2}{3}} + \rho_1{}^{-\frac{2}{3}} = (2a)^{-\frac{2}{3}}.$$

**18. If ρ be the radius of curvature at any point P on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, and CD is the semi-diameter conjugate to CP, then $\rho = CD^3/ab$.

19. Find the locus of the centres of curvature at different points on a conic in the following cases :—[a]

$$\text{(i)} \quad y^2 = 4ax. \quad \text{(ii)} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

[a] The locus of the centres of curvature at different points on a conic (or any curve) is called the *evolute* of the conic (or the curve). The original curve is called the *involute* of the locus thus found (which is its evolute).

**53.

ON SIMILAR CONICS.

53·1. Definitions.

Two conics $S=0$ and $S'=0$ (lying in the same plane) are said to be *similar and similarly situated*, if two points O and O' can be found in their plane such that any radius vector drawn from O to $S=0$ bears a constant ratio to the parallel radius vector drawn from O' to $S'=0$. Thus, two *concentric circles* may be said to form a pair of such conics, both O and O' coinciding with their common centre.

If, however, two points O and O' can be found such that any radius vector OP drawn from O to $S=0$ bears a constant ratio to the radius vector $O'P'$ drawn from O' to $S'=0$ making a constant angle with the direction OP, then the two conics are said be *similar*. [21] From this it is clear that such a pair conics can be made *similar and similarly situated* by rotating one of them through a proper angle about a point in their plane.

53·2. To find the condition that a pair of central conics may be similar and similarly situated.

Let the conics be given by $S=0$ and $S'=0$, where S and S' are as in Art. 52.

Now, changing to parallel axes through the centre of the first as the origin, their equations can be written as

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[21] Such a pair of conics are *not* necessarily similarly situated.

$$ax^2 + 2hxy + by^2 = 1 \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

and $a'x^2 + 2h'xy + b'y^2 + 2g'x + 2f'y + c' = 0, \quad \ldots \quad (2)$

a, h, b, a', h', b' being the same as before.[22]

Hence, if r be the length of the semi-diameter making an angle θ with the axis of x, we have

$$\frac{1}{r^2} = a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta.$$

Again, changing (1) and (2) to parallel axes through the centre of the second as the origin, it can be easily seen that the length r_1 of the parallel semi-diameter of (2) is given by

$$\frac{1}{r_1{}^2} = a'\cos^2\theta + 2h'\cos\theta\sin\theta + b'\sin^2\theta.$$

But a mere translation of axes cannot change the terms of second degree in (1).[23] Hence, in order that the conics may be similar and similarly situated, we must have

$$\frac{a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta}{a'\cos^2\theta + 2h'\cos\ \sin\theta + b'\sin^2\theta} = \text{ a constant for}$$

all values of θ.

But this cannot be unless $\dfrac{a}{a'} = \dfrac{h}{h'} = \dfrac{b}{b'}$.

These are, therefore, the required conditions.

53·21. Note on the above.

From the conditions obtained above, it is clear that the *asymptotes* of $S = 0$ are parallel to those of $S' = 0$, and conversely. And, as the axes bisect the angles between

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[22] See Art. 4·31.

the asymptotes, the axes of $S=0$ are parallel to those of $S'=0$.

53·3. To find the condition that two conics may be similar.

If the conics are similar, they can be made similarly situated (if not already so) by turning one of them through a proper angle about a point in its plane. Hence, the angle between the asymptotes of the one must be equal to the angle between those of the other.

$$\therefore \quad \frac{ab-h^2}{a+b} = \frac{a'b'-h'^2}{a'+b'}, \text{ by (3) of Art. 13}$$

Examples.

1. Show that two parabolas which have their axes in the same direction are similar and similarly situated.

2. Two conics which have their eccentricities equal are similar.

**54.

ON CONFOCAL CONICS

54·1. Definition and general equation of confocal conics.

A group of central conics are said to form *a confocal system* when they have the same foci. Now, all conics having the same foci must have the same centre and axes. Hence, the equation of any conic confocal with the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is given by $\dfrac{x^2}{a^2+\lambda}+\dfrac{y^2}{b^2+\lambda}=1$, [23] ... ... (1)

where λ is a variable parameter.

54·2. Some geometrical theorems.

(i) *Through any point in the plane of an ellipse two confocal conics can be drawn; of these one is an ellipse and the other is a hyperbola.*

Let the equation any conic confocal with the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ be $\dfrac{x^2}{a^2+\lambda}+\dfrac{y^2}{b^2+\lambda}=1$; and let $P\equiv(h, k)$ be the given point.

Now, if the confocal conic passes through P, we must have

$$\frac{h^2}{a^2+\lambda}+\frac{k^2}{b^2+\lambda}=1.$$

$$\therefore\quad (a^2+\lambda)(b^2+\lambda)-h^2(b^2+\lambda)-k^2(a^2+\lambda)=0. \quad \dots \quad (2)$$

This being a quadratic equation in λ, it is clear that two conics confocal with the ellipse (or central conic) passes through any point in the plane of the ellipse.

Again, putting $\lambda=+\infty$ in (2), we see that the expression on the left-hand side of (2) becomes *positive.* [24]

Similarly, it can be seen that the above expression become *negative* or *positive* according as we substitute $-b^2$ or $-a^2$ for λ.

[23] For, $CS=ae=\sqrt{a^2-b^2}=\sqrt{(a^2+\lambda)-(b^2+\lambda)}$ *for all values of* λ.

[24] For, the left-hand side of (2) $=(b^2+\lambda)\ \{(a^2+\lambda)-h^2-k^2 \times(1+a^2/\lambda)/(1+b^2/\lambda)\}\ =\ \infty\ \{\infty-h^2-k^2\}=\infty=$positive.

Hence, by a well-known theorem of the Theory of Equations,[25] one root of (2) in λ lies between ∞ and $-b^2$, and the other root lies between $-b^2$ and $-a^2$. Hence, one value of λ will make both $a^2+\lambda$ and $b^2+\lambda$ positive, and the other will make $a^2+\lambda$ positive and $b^2+\lambda$ negative.

Hence, from (1), it is clear that one of the confocals through (h, k) is an *ellipse* and the other is a *hyperbola*. Hence the above results.

(ii) Confocal conics cut at right angles at every point of their intersection.

Let the equations of the conics which pass through the point (h, k) and are confocal with the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

be $\frac{x^2}{a^2+\lambda_1}+\frac{y^2}{b^2+\lambda_1}=1$ and $\frac{x^2}{a^2+\lambda_2}+\frac{y^2}{b^2+\lambda_2}=1$,

where λ_1 and λ_2 are the roots of (2) in λ.

Now, it can be easily seen that the 'm' of the tangent at (h, k) to the first of these confocals $=-\frac{h(b^2+\lambda_1)}{k(a^2+\lambda_1)}$; and that the '$m$' of the tangent at (h, k) to the second $=-\frac{h(b^2+\lambda_2)}{k(a^2+\lambda_2)}$. Hence, the condition that these confocals may cut at right angles is

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[25] The theorem may be stated as follows:—

If two real quantites a and b are substituted for x in $f(x)$, and we get results with different signs, then there is one (or an odd number of) roots of $f(x)=0$ between a and b.

$$\frac{h^2(b^2+\lambda_1)(b^2+\lambda_2)}{k^2(a^2+\lambda_1)(a^2+\lambda_2)} = -1. \quad \ldots \quad \ldots \quad (3)$$

Again, since the point (h, k) lies on each of the confocals, we must have

$$\frac{h^2}{a^2+\lambda_1} + \frac{k^2}{b^2+\lambda_1} = 1 \text{ and } \frac{h^2}{a^2+\lambda_2} + \frac{k^2}{b^2+\lambda_2} = 1.$$

$$\therefore \quad \frac{h^2}{(a^2+\lambda_1)(a^2+\lambda_2)} + \frac{k^2}{(a^2+\lambda_1)(b^2+\lambda_2)} = 0, \text{ by sub-}$$

traction.

And as it is easy to see that this is the same as (3), the truth of the theorem is establised.

Examples.

If ϕ be the angle between the tangents from (h, k) to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then $\tan\frac{\phi}{2} = \sqrt{-\frac{\lambda_2}{\lambda_1}}$, where λ_1 and λ_2 are the parameters of the confocals through (h, k).[26]

2 Show that one (and only one) conic confocal with the ellipse of Ex. 1 can be drawn to touch a given line.

**3. Show that the equation of the pair of tangents from (h, k) to the ellipse of Ex. 1 may be written in the form $\frac{x_1^2}{\lambda_1}+\frac{y_1^2}{\lambda_2} = 0$, where λ_1 and λ_2 are the parameters of the confocals through (h, k).

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26 See Ex. 1 of Art. 38.

**4. If the conics $ax^2+2hxy+by^2=1$ and $a'x^2+2h'xy+b'y^2=1$ are placed so as to be confocal, then

$$\frac{(a-b)^2+4h^2}{(ab-h^2)^2}=\frac{(a'-b')^2+4h'^2}{(a'b'-h'^2)^2}.$$

**5. Find the co-ordinates of any point in the plane of an ellipse in terms of the parameters of the confocals through it.

6. Find the locus of the pole of a given straight line with respect to a system of confocal conics.

7. Find the locus of points such that tangents drawn from them, one to each of two given confocal conics, may be at right angles to each other.